On rapidly varying sequences

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Mutually asymptotic inverse functions

Real functions $f, g : [a, +\infty) \longmapsto \mathbb{R}$, $(a > 0)$, are mutually asymptotic inverse, in denotation

$$f(x) \overset{*}{\sim} g(x), \text{as } x \to +\infty,$$

(see e.g. [1, 4, 6]), if for each $\lambda > 1$, there is an $x_0 = x_0(\lambda) \geqslant a$ such that the inequality

$$f\left(\frac{x}{\lambda}\right) \leqslant g(x) \leqslant f(\lambda x), \tag{1}$$

is satisfied for each $x \geqslant x_0$.

Mutually rapidly equivalent functions

Specially, real functions (which are mutually asymptotic inverse) $f, g : [a, +\infty) \longmapsto (0, +\infty)$, $(a > 0)$, are mutually rapidly equivalent, in denotation

$$f(x) \overset{r}{\sim} g(x), \text{as } x \to +\infty,$$

(see e.g.[7, 10]), if

$$\lim_{x \to +\infty} \frac{f(\lambda x)}{g(x)} = +\infty \tag{2}$$

and

$$\lim_{x \to +\infty} \frac{g(\lambda x)}{f(x)} = +\infty \tag{3}$$

hold for each $\lambda > 1$.

Mutually rapidly equivalent sequences

Sequences of positive real numbers $(c_n)_{n\in\mathbb{N}}$ and $(d_n)_{n\in\mathbb{N}}$ are mutually rapidly equivalent in denotation

$$c_n \overset{r}{\sim} d_n, \text{as } n \to +\infty,$$

(see e.g.[7]), if

$$\lim_{n\to+\infty} \frac{c_{[\lambda n]}}{d_n} = +\infty \tag{4}$$

and

$$\lim_{n\to+\infty} \frac{d_{[\lambda n]}}{c_n} = +\infty \tag{5}$$

hold for each $\lambda > 1$.

Real, measurable function $f : [a, +\infty) \longmapsto (0, +\infty),\ (a > 0)$ is rapidly varying in the sense of de Haan (see e.g. [1]), if

$$\lim_{x \to +\infty} \frac{f(\lambda x)}{f(x)} = +\infty, \tag{6}$$

holds for each $\lambda > 1$.
The set of all these functions is denoted by $R_{\infty,f}$.
The class $R_{\infty,f}$ is very important in asymptotic analysis (see e.g. [1, 7]).

Rapidly varying sequences $R_{\infty,s}$

A sequence $c = (c_n)_{n\in\mathbb{N}}$ of positive real numbers is said to be rapidly varying in the sense of de Haan (see e.g. [2]), if

$$\lim_{n\to+\infty} \frac{c_{[\lambda n]}}{c_n} = +\infty, \tag{7}$$

holds for each $\lambda > 1$.
The set of all these sequences is denoted by $R_{\infty,s}$.
The class $R_{\infty,s}$ is very important in asymptotic analysis, game theory and selection principles theory (see e.g. [2, 3, 5]).

In this paper the set of all positive real sequences will be denoted with $\mathbb{S}$, and the set of all nondecreasing elements from $\mathbb{S}$ will be denoted with $\mathbb{S}_1$.

Proposition 1.

Let sequences $c = (c_n)_{n\in\mathbb{N}}$ and $d = (d_n)_{n\in\mathbb{N}}$ be elements from $\mathbb{S}_1$. If $c_n \overset{r}{\sim} d_n$ as $n \to +\infty$, then $c \in R_{\infty,s}$ and $d \in R_{\infty,s}$.

Proposition 2.

Relation $\overset{r}{\sim}$ is an equivalence relation in $R_{\infty,s} \cap \mathbb{S}_1$.

Let $c = (c_n)_{n\in\mathbb{N}}$ be element from $\mathbb{S}$. Then (see e.g. [1])

$$\overline{c} = (\overline{c}_n)_{n\in\mathbb{N}}, \overline{c}_n = \max\{c_p | 1 \leqslant p \leqslant n\} (\text{cumulative minimum}) \tag{8}$$

and

$$\underline{c} = (\underline{c}_n)_{n\in\mathbb{N}}, \underline{c}_n = \min\{c_p | p \geqslant n\} (\text{cumulative maximum}) \tag{9}$$

are called upper and lower associate of the sequence c.

Proposition 3.

Let $c \in R_{\infty,s}$ and $c_n \asymp d_n$, as $n \to +\infty$ ($\asymp$ is the weak asymptotic equivalence relation, see e.g. [1]), then $c_n \overset{r}{\sim} d_n$, as $n \to +\infty$.

Proposition 4.

Let $c \in \mathbb{S}$. Then the sequence $c \in R_{\infty,s}$ if and only if $\underline{c}_n \overset{r}{\sim} \overline{c}_n$ holds, as $n \to +\infty$.

Proposition 5.

Let $c \in \mathbb{S}$ and let the sequence $d = (d_n)_{n \in \mathbb{N}}$ be an element of the class $R_{\infty,s}$. If $\underline{d}_n \leqslant c_n \leqslant \overline{d}_n$, for $n \geqslant n_0 \geqslant 1$, then $c \in R_{\infty,s}$.

Let $c = (c_n)_{n\in\mathbb{N}} \in \mathbb{S}$, then (see e.g. [7])

$$\underset{\sim}{c} = (\underset{\sim}{c_n})_{n\in\mathbb{N}}, \ \underset{\sim}{c_n} = \frac{1}{n}\sum_{k=1}^{n-1} c_k \tag{10}$$

is called the sequence of additive midpoint of the sequence c (see e.g. [1]).

Proposition 6.

Let $c = (c_n)_{n\in\mathbb{N}} \in \ R_{\infty,s}$, then $\underset{\sim}{c} \in R_{\infty,s}$ and $c_n \sim \underset{\sim}{c_n}$, as $n \to +\infty$ ($\sim$ is the strong asymptotic equivalence, see e.g. [1]).

Let us state the definition of well known selection principles, which we call Kocinac's α_i selection principles (see e.g. [8]).

Definition 1.

Let $\mathcal{A}$ and $\mathcal{B}$ be subfamilies of the set $\mathbb{S}$. The symbol $\alpha_i(\mathcal{A}, \mathcal{B})$, $i \in \{2, 3, 4\}$, denotes the following selection hypotheses:

1. $\alpha_2(\mathcal{A}, \mathcal{B})$: *for each sequence $(A^n)_{n\in\mathbb{N}}$ of elements from $\mathcal{A}$, there is an element $b \in \mathcal{B}$, so that $b \cap A^n$ is infinite for each $n \in \mathbb{N}$;*
2. $\alpha_3(\mathcal{A}, \mathcal{B})$: *for each sequence $(A^n)_{n\in\mathbb{N}}$ of elements from $\mathcal{A}$, there is an element $b \in \mathcal{B}$, so that $b \cap A^n$ is infinite for infinitely many $n \in \mathbb{N}$;*
3. $\alpha_4(\mathcal{A}, \mathcal{B})$: *for each sequence $(A^n)_{n\in\mathbb{N}}$ of elements from $\mathcal{A}$, there is an element $b \in \mathcal{B}$, so that $b \cap A^n$ is nonempty for infinitely many $n \in \mathbb{N}$.*

Let us observe the definition of an infinitely long game related to α_2 (see e.g. [8, 9]).

Definition 2.

Let $\mathcal{A}$ and $\mathcal{B}$ be nonempty subfamilies of the set $\mathbb{S}$. The symbol $G_{\alpha_2}(\mathcal{A}, \mathcal{B})$ denotes the following infinitely long game for two players, I and II, who play a round for each natural number n. In the first round the player I plays an arbitrary element $A^1 \in \mathcal{A}$, and the player II chooses a subsequence $A^{r_1(j)}, j \in \mathbb{N}$, of the sequence A^1. At the k^{th} round, $k \geqslant 2$, the player I plays an arbitrary element $A^k \in \mathcal{A}$ and the player II chooses a subsequence $A^{r_k(j)}$ of the sequence A^k, such that $A^{r_k(j)} \cap A^{r_p(j)} = \emptyset$ is satisfied, for each $p \leqslant k-1$. The player II wins the play $A^1, A^{r_1(j)}; \ldots ; A^k, A^{r_k(j)}; \ldots$ if and only if all elements from $Y = \cup_{k \in \mathbb{N}} \cup_{j \in \mathbb{N}} A^{r_k(j)}$ form a subsequence $y \in \mathcal{B}$.

Let $c = (c_n)_{n\in\mathbb{N}} \in \mathbb{S}$. Then we define

$$[c]_r = \{d = (d_n)_{n\in\mathbb{N}} \in \mathbb{S} |\ c_n \overset{r}{\sim} d_n,\ n \to +\infty\} \qquad (11)$$

in $R_{\infty,s}$, and for $c = (c_n)_{n\in\mathbb{N}} \in \mathbb{S}_1$ we define

$$[c]_r' = \{d = (d_n)_{n\in\mathbb{N}} \in \mathbb{S}_1 |\ c_n \overset{r}{\sim} d_n,\ n \to +\infty\} \qquad (12)$$

as the class of equivalence in $R_{\infty,s} \cap \mathbb{S}_1$, according to Propositions 1 and 2 .

Proposition 7.

The second player has a winning strategy in the game $G_{\alpha_2}([c]_r^{\prime}, [c]_r)$, *for each fixed element* $c \in R_{\infty,s} \cap \mathbb{S}_1$.

Corollary 1.

The selection principle $\alpha_2([c]_r^{\prime}, [c]_r)$ *holds for each fixed element* c *from the class* $R_{\infty,s} \cap \mathbb{S}_1$.

Note 1.

From the Corollary 1, and [9], the selection principles $\alpha_i([c]_r^{\prime}, [c]_r)$ *hold for* $i \in \{3, 4\}$, *where* c *is arbitrary fixed element from* $R_{\infty,s} \cap \mathbb{S}_1$.

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THANK YOU!