Some results on Mean Field Games with (strong) aggregation

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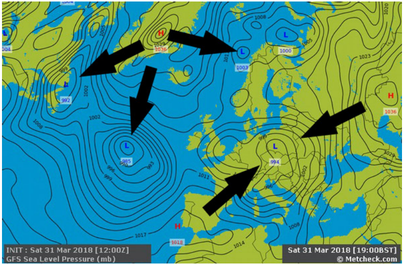
joint work with M. Cirant (Università di Padova)

From statistical physics

Inspired by physics of particles.

- Statistical idea: use a mean (mean field) to study the interaction between particles.
- Mean field representative of the particles as a whole.
- Example: pressure of the air.



Strategical interaction between agents in economy.

Theory of general equilibrium in economy: Mean field=vector of prices. Clear approach of what the market is.

Game theory: solution of a MFG describes Nash equilibria when the number of players $N \to \infty$.

Aims:

- A “mean field” to have a relevant representation of reality.
- Obtain a macroscopic model (continuum) of ”mean field” type simpler than the discrete model with N agents.

Basic purpose of MFG

Describe interacting "rational" particles (players)

- players have controlled dynamical state ↔ (stochastic) diff. equation
- players are indistinguishable ↔ mean-field interaction: no privileged interaction with some particles
- players try to minimize a cost functional

Study equilibria in the large population limit, i.e. as the number N of players $\to \infty$

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Mean field games-applications

Economy:

- economical equilibria with rational expectations
- financial markets (formation of prices and dynamical equilibria, formation of volatility)
- game theory
- environmental policy

Engineering:

- control of wireless power, management of demand in electrical power,
- traffic problems

Social sciences:

- crowd motion (mexican wave, pedestrian dynamic, congestion...)
- opinion dynamic and consensus problems,
- models of distribution of the population (e.g. segregation).

The limit problem

Reduce the asymptotic analysis to one typical player that interacts with a population with density $\mathbf{m}(\mathbf{x}, \mathbf{t}) : \mathbb{R}^d \times [0, T] \to \mathbb{R}$.

State of a typical player: $X_0 \sim m_0$,

$$dX_s = v_s ds + \sqrt{2}\, dB_s \qquad \text{in } \mathbb{R}^d,$$

v_s is the controlled velocity, B_s a Brownian motion, with cost:

$$J_{\mathbf{m}}(v) = \mathbb{E} \int_0^T \frac{1}{2}|v_s|^2 + V(X_s) + f\big(\mathbf{m}(X_s)\big)\, ds + u_T(X_T),$$

- $V : \mathbb{R}^d \to \mathbb{R}$ encodes "spatial preferences"
- $f : \mathbb{R} \to \mathbb{R}$ is the "coupling" (could be a functional over $\mathscr{P}(\mathbb{R}^d)$)
- $u_T : \mathbb{R}^d \to \mathbb{R}$ is the final cost

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MFG SYSTEM

Suppose that, given $\mathbf{m}$, the typical player behaves optimally. Denote by μ her density. In an equilibrium / consensus regime,

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PDE description:

$$\begin{cases} -\partial_t u - \Delta u + \frac{1}{2}|\nabla u|^2 = f\Big(m(x,t)\Big) + V(x) & \text{in } \mathbb{R}^d \times (0,T) \\ \partial_t m - \Delta m - \operatorname{div}(m\nabla u) = 0 & \text{in } \mathbb{R}^d \times (0,T) \\ m(0) = m_0, \quad u(T) = u_T & \text{in } \mathbb{R}^d \end{cases} \tag{MFG}$$

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Hamilton-Jacobi eq. comes from the DPP: best player's strategy. $u(x,t)$ is the value function of the typical player, i.e.

$$u(x,t) = \inf_v \mathbb{E} \int_t^T \frac{1}{2}|v_s|^2 + V(X_s) + f\big(\mathbf{m}(X_s)\big)\, ds + u_T(X_T)$$

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PDE description: The (forward) MFG backward-forward system

$$\begin{cases} -\partial_t u - \Delta u + \frac{1}{2}|\nabla u|^2 = f\big(m(x,t)\big) + V(x) & \text{in } \mathbb{R}^d \times (0,T) \\ \partial_t m - \Delta m - \operatorname{div}(m\nabla u) = 0 & \text{in } \mathbb{R}^d \times (0,T) \\ m(0) = m_0, \quad u(T) = u_T & \text{in } \mathbb{R}^d \end{cases} \tag{MFG}$$

Fokker-Planck equation: (optimal) distribution of the typical player / population.
The drift $-\nabla u$ is the optimal strategy.

Connections with the large population problem

In the N-players game, interactions are with $\frac{1}{N}\sum_i \delta_{X_t^i}$ (instead of m)

Players look for a consensus - Nash equilibrium (i.e. no interest for any player to leave the consensus):

Described by huge Nash system of N quasilinear (HJB) equations.

Lasry - Lions $\approx$ '06	$\longleftrightarrow$	Nash equilibria $\xrightarrow{N\to\infty}$ solution to (MFG)
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A variational viewpoint

MFG system has a fixed point structure: $\mathbf{m} \mapsto \mu \equiv \mathbf{m}$.

Anyhow, min. problem for the single player can be recast into

$$J_{\mathbf{m}}(v,\mu) = \iint_{\mathbb{R}^d \times (0,T)} \frac{1}{2}|v|^2\mu + V\mu + f(\mathbf{m})\mu \, dxdt + \int_{\mathbb{R}^d} u_T(\mu(T))dx$$

subject to the Fokker-Planck constraint

$$\partial_t \mu - \Delta\mu - \mathrm{div}(v\mu) = 0.$$

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These couples can be found by minimization of

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Reminiscent of Benamou–Brenier dynamic formulation of optimal transport.

Developed in the context of MFG by several authors: Cardaliaguet, Carlier, Meszaros, Santambrogio, Silva, ...

Indicates the crucial role played by

$$\text{(incr.) monotonicity of } f \longleftrightarrow \text{convexity of } F.$$

In this case one has

- existence and uniqueness of weak solutions,
- stability: for $\delta \in (0,1)$, $m(x, \delta T) \to \bar{m}(x)$ as $T \to \infty$,
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Existence of classical solutions

- smallness conditions, perturbative regime:

$$\begin{cases} -\partial_t u - \Delta u = -\varepsilon H(x,t,\nabla u, m) & \text{in } \mathbb{R}^d \times (0,T) \\ \partial_t m - \Delta m = -\varepsilon \operatorname{div}(m \nabla_p H(x,t,\nabla u, m)) & \text{in } \mathbb{R}^d \times (0,T) \\ m(0) = m_0, \quad u(T) = u_T & \text{in } \mathbb{R}^d \end{cases}$$

solutions exist provided that m_0, u_T are "small", or T is small, or ε is small [Ambrose, C.-Gianni-Mannucci, ...]

- monotone case: f increasing (competition),
  - $H = |\nabla u|^2 - f(m)$, no restrictions on f
  - $H = |\nabla u|^\gamma - f(m)$, $\gamma \neq 2$ needs "mild growth", that is $f \sim m^\alpha$,

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  In the model case $f \sim -m^\alpha$, classical solutions exist if

$$\alpha < \frac{2}{d}.$$

Indeed, in such a regime, the energy functional

$$\mathcal{E}(v,\mu) = \iint_{\mathbb{R}^d \times (0,T)} \frac{1}{2}|v|^2\mu + V\mu - \mu^{\alpha+1}\, dxdt + \int_{\mathbb{R}^d} u_T(\mu(T))dx,$$

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$$\left(\iint_{\mathbb{R}^d\times(0,T)} \mu^{\alpha+1}\right)^{\delta} \lesssim \iint_{\mathbb{R}^d\times(0,T)} |v|^2 \mu$$

and $\delta < 1$ if and only if $\alpha < \frac{2}{d}$ (while $\delta > 0$ if and only if $\alpha < \frac{2}{d-2}$)

Some phenomena that arise:

- No uniqueness
- periodic in time solutions and travelling waves
- concentration phenomena as the viscosity vanishes

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Main results

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V be a bounded potential (not too much radially decreasing).

Theorem (non-existence)

Let $\alpha \geq \frac{2}{d}$. Under some integrability assumptions on m_0, u_T, there exist σ^*, T^* such that if

$$\sigma > \sigma^* \qquad \text{and} \qquad T > T^*,$$

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Theorem (existence)

Let $\frac{2}{d} \leq \alpha < \frac{2}{d-2}$. Under some integrability assumptions on m_0, u_T, there exists $\sigma_0 = \sigma_0(d, \alpha, m_0, u_T, T\|\Delta V\|_\infty)$ such that if

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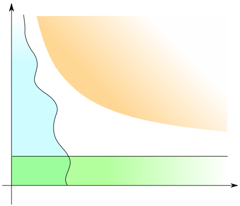
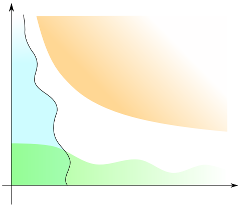
then the MFG system has at least a (classical) solution.





$V \equiv 0$, $V \neq 0$.

short-time existence, small-cost existence, non-existence

On the Planning problem

Consider the problem

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NON-EXISTENCE

Based on three ingredients:

(i) conservation of energy

(ii) momentum

(iii) "virial identity"

} involve $\int_{\mathbb{R}^d} |x|^2 \, m(x,t)$

(i) the MFG system can be seen as an Hamiltonian system:

$$\begin{cases} u_t = \frac{\delta \mathcal{H}}{\delta m}(m,u) \\ m_t = -\frac{\delta \mathcal{H}}{\delta u}(m,u) \end{cases}$$

where

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obtained by $HJ \cdot \partial_t m + FKP \cdot \partial_t u$

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(ii) testing FKP by x^2,

$$\frac{d}{dt}\int_{\mathbb{R}^d} m(t)x^2\,dx = 2d - \int_{\mathbb{R}^d} m(t)\nabla u(t)\cdot x\,dx$$

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$$\frac{d}{dt}\int_{\mathbb{R}^d} m(t)\nabla u(t)\cdot x\,dx = -2\int_{\mathbb{R}^d} \nabla m\cdot\nabla u + \frac{1}{2}|\nabla u|^2 m\,dx - \frac{\alpha d}{\alpha+1}\int_{\mathbb{R}^d} m^{\alpha+1} - \nabla V\cdot xm\,dx.$$

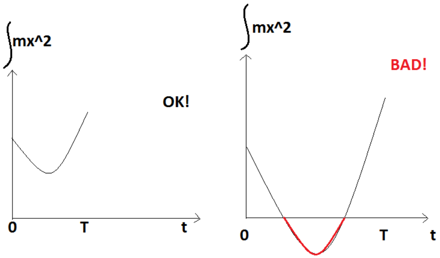
Then,

$$\frac{d^2}{dt^2}\int_{\mathbb{R}^d} m(t)x^2\,dx = 4E + \overbrace{\frac{2d\sigma}{\alpha+1}\left(\alpha-\frac{2}{d}\right)\int_{\mathbb{R}^d} m^{\alpha+1}}^{\geq 0} + \overbrace{2\int_{\mathbb{R}^d}[2(V-\inf V)+\nabla V\cdot x]m\,dx}^{\geq 0}$$
$$\geq 4E$$

$$\Rightarrow \quad \frac{d^2}{dt^2}\int_{\mathbb{R}^d} m(t)x^2\,dx \geq + \int_{\mathbb{R}^d} -\frac{|\nabla m_0|^2}{m_0} - Vm_0 + \sigma m_0^{\alpha+1}\,dx > 0$$

whenever σ is large (or m_0 is "large").

Planning problem: $\int_{\mathbb{R}^d} m_0 x^2 dx$, $\int_{\mathbb{R}^d} m_T x^2 dx$ are given.



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MFG problem: $\int_{\mathbb{R}^d} m_0 x^2 dx$ is given, $\frac{d}{dt}\int_{\mathbb{R}^d} m(T)x^2 dx \leq 2d$.

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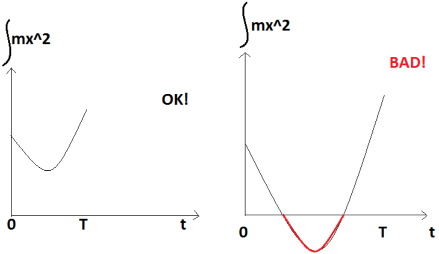

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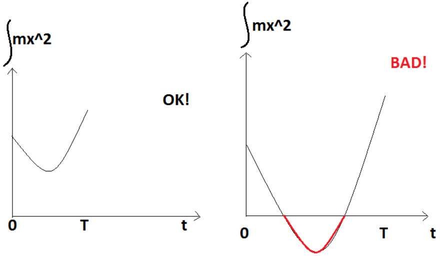

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Existence

As usual, solutions can be obtained as fixed points of the operator $m \mapsto \mu$,

$$\begin{cases} -\partial_t u - \Delta u + \frac{1}{2}|\nabla u|^2 = -\sigma m^\alpha + V & \text{in } \mathbb{R}^d \times (0,T) \\ \partial_t \mu - \Delta \mu - \operatorname{div}(\mu \nabla u) = 0 & \text{in } \mathbb{R}^d \times (0,T) \\ \mu(0) = m_0, \quad u(T) = u_T & \text{in } \mathbb{R}^d \end{cases}$$

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$$||m||_X \leq C.$$

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A PRIORI ESTIMATES IN $L^{2\alpha+1}$

Let $\frac{2}{d} \leq \alpha \leq \frac{2}{d-2}$ and (u, m) be a solution to the MFG system. Then,

$$\frac{Y^\delta}{C} - \sigma^2 Y \leq 1, \quad \delta < 1, \quad Y = \left(\int_0^T \int_{\mathbb{R}^d} \mu^{2\alpha+1}(t)\, dxdt \right)^2,$$

where C depends on d, α, m_0, u_T and δ depends on α.
Enough to set up a fixed point procedure (σ small).
(combination of conservation of energy, “Second order estimates” and parabolic regularity)

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  - non-existence:
    (i), (ii), (iii) hold, but not obvious how to combine them
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Thank you for your attention.