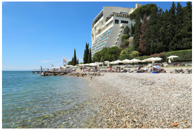
Minimal triangulations

from **Wacław Marzantowicz** at

to **8 ECM**





June 25, 2021

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The classical theorem (Whithney, Cairns): $\forall$ smooth manifold posses a triangulation, i.e. $\exists$ a simplicial com. K such that $M \underset{homeo}{\simeq} |K|$, where $|K|$ is the body of K, i.e. its geometrical realization.
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For many reasons, **it is important to know (estimate) the least number of vertices, or simplices in general, which is necessary to triangulate M.**

Let K be a finite simplicial complex of dimension d.
Associate with K a $d+1$ vector with
non-negative integral coordinates.

$$\overrightarrow{f}(K) := (f_0(K), f_1(K), \dots, f_d(K)), \quad \text{with} \quad f_i = \#\{\sigma^i \in K\}$$

where σ^i is a simplex of dim $= i$.

Definition

We say that a triangulation K of M is vertex minimal, or shortly minimal if $f_0(K) = \min f_0(L)$, where L is a triangulation of M.
We say that a triangulation K of M is totally minimal if $\Sigma_{i=0}^d f_i(K) = \min \Sigma_{i=0}^d f_i(L)$, where L is a triangulation of M.

Problem

For given manifold M

- U: Find an upper estimate of $f_0(K)$, respectively $\Sigma_{i=0}^{d} f_i(K)$, where K is the minimal, respectively totally minimal triangulation of M
- L: Find a lower estimate of $f_0(K)$, respectively $\Sigma_{i=0}^{d} f_i(K)$, where K is the minimal, respectively totally minimal triangulation of M
- E: Find the value of $f_0(K)$, respectively $\Sigma_{i=0}^{d} f_i(K)$, where K is the minimal, respectively totally minimal triangulation of M

It is apparent that we would study only the case when M is connected.

- $\dim M = 1$. Then either $M = S^1$ is the circle, or $M = I = [0,1]$ is the interval, depending whether it without or with the boundary. It is evident that the minimal triangulation of S^1 consists of three 1 simplices and thus of three vertices, and $I = \sigma^1$ forms a minimal triangulation of itself.
- $M = S^d$, d-dimensional sphere, $d \geq 2$. Observing that S^d is homeomorphic to $\partial\Delta^{d+1}$ the boundary of $d+1$-simplex, we have a triangulation of S^d with $f_i(S^d) = \binom{d+2}{i+1}$ i-dimensional simplices for $0 \leq i \leq d$, thus $f_0(S^d) = d+2$, which gives the upper estimate of $f_i(S^d)$, e.g. $f_0(S^d) \leq d+2$. On the other hand we will show later that $f_0(S^d) \geq d+2$ by a topological argument. Consequently, the above is a minimal triangulation.

Closed surfaces, i.e. compact 2-manifolds without boundary.

Theorem (2)

(Jüngerman-Ringel) Let S_g be a closed surface different from the orientable surface of genus 2 (M2), the Klein bottle (N2) and the non-orientable surface of genus 3 (N3). $\exists$ a triangulation of S_g with n vertices iff

$$(+) \qquad n \geq \frac{7+\sqrt{49-24\chi(S_g)}}{2}.$$

For cases (M2), (N2) and (N3), replace n by $n-1$ in the formula, denoted by $n(S_g)$, or shortly n_g.

If $K \sim S_g$

$$\chi(S_g) = \chi(K) = f_0 - f_1 + f_2$$

$$f_2 \geq \tfrac{3}{2} f_1 \text{ and } f_1 \leq \binom{f_0}{2} = \frac{f_0(f_0-1)}{2}$$

Consequently, we get

$$(++) \qquad 6\chi(S_g) = \chi(K) \geq 6f_0 - f_0(f_0-1).$$

It is easy to check that the boundary (minimal positive) integral solution of the inequality $(++)$ is the same (with identification $f_0 = n(S_g)$) as this of $(+)$.

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The original proof of Theorem [21] is given by an effective but a little bit pedestrian construction of a triangulation K of S_g with $f_0(K) = n_g$ that obviously gives the upper estimate of $f_0(S_g)$ by n_g.

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In next we shall present recent result of Borghini and Miniam which gives a topological proof of the inequality $n_g \leq f_0(S_g)$.

Remark

To prove inequality $(+)$, Jungerman and Ringel also used a simplicial embedding of $\mathbb{K}_{[n_g]}$ into S_g as a graph consisting of edges of any triangulation of S_g.

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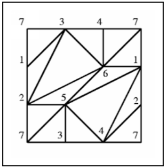
3. $f_0 = 8$ *if* $S_g = \mathcal{K}$ *the Klein bottle,* $f_0 = 6$ *if* $S_g = \mathbb{R}P^2$ *the projective plane*
4. $f_0 = 9$ *if* $S_g = N3$ *the non-oriented surface of genus* $g = 3$.

Remark

Some authors, by the minimal triangulation mean a triangulation of a d-manifold M, with minimal number of simplices of the highest dimension d.

Examples



Minimal triangulation of the torus, $f_0 = 7$.

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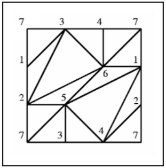

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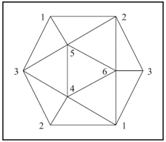

Minimal triangulation of the projective plane, $f_0 = 6$.

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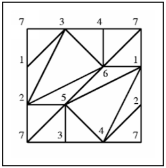

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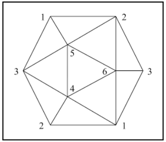

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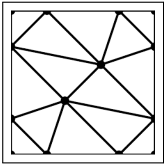

Minimal triangulation of the Klein bottle, $f_0 = 8$.

For a complete and exhaustive review of classical results, we refer to the survey articles of

- B. Datta: *Minimal triangulations of manifolds*, arXiv:math/0701735.
- F. Lutz: *Triangulated Manifolds with Few Vertices: Combinatorial Manifolds*, arXiv:math/0506372.

In [24] Kühnel and Lassmann showed that the d-dimensional torus $\mathbb{T}$ can be triangulated combinatorially with $2^{d+1}-1$ vertices (for $d \geq 2$).
They posed the **conjecture:** the triangulation of d-torus with $2^{d+1}-1$ vertices is minimal.

Chromatic number:

Assigning to each region (country) of a map of $M = S_g$ a vertex and to each border an edge we replace the "problem of coloring of map" drawn on the surface S_g by the problem of finding of the chromatic number $\mathrm{chr}(\Gamma)$ of the associated graph.

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In this way, we obtain, within a unified framework, several estimates which are either new or extensions of results that have been previously obtained by ad hoc combinatorial arguments. Moreover, our methods give results that are valid for entire homotopy classes of spaces.

Definitions

Definition (Karoubi-Weibel, cf. [23])

- $\{U_i\}$ is a good cover of X if $U_{i_1} \cap U_{i_2} \cap \cdots \cap U_{i_n} \neq \emptyset \implies U_{i_1} \cap U_{i_2} \cdots \cap U_{i_n}$ is contractible.

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Examples: $\mathrm{ct}(X) = 1 \Leftrightarrow X$ contractible.
$\mathrm{ct}(X) = 2 \Leftrightarrow X$ disjoint union of two contractible sets.

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- [23]: $n + 2 \leq \mathrm{ct}(\mathbb{R}P^n) \leq 2m + 3$ wrong,
  we show right estimate from below.
- the Hawaiian earring X does not admit any good covers, i.e $\mathrm{sct}(X) = \infty$.

Definition

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  i.e. $\Delta^{PL}(M) = f_0(K)$, K - the minimal triangulation
- If X has the homotopy type of compact polyhedron we introduce a homotopy analogue of $\Delta(P)$ as
  $\Delta^{\simeq}(X) := \min\{\Delta(P) \mid P \simeq X\}$.

Computing $\Delta(P)$ and its variants is a hard and intensively studied problem of combinatorial topology - see Datta [12] and Lutz [25] for surveys of the vast body of work related to this question. Clearly, $\Delta^{\simeq}(P)$ is a lower bound for other invariants, since

$$\Delta^{\simeq}(P) \leq \Delta(P), \text{ and if } M \text{ is a PL-manifold } \Delta^{\simeq}(M) \leq \Delta^{PL}(M).$$

Theorem (9 .)

If X has the homotopy type of a finite polyhedron, then

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Open problem (GMP - 2017)

For which closed PL-manifolds M we have
$\mathrm{ct}(M) = \Delta^{PL}(M) = f_0(M)$*?*

In general the equality is false already for surfaces - Borghini & Minian (surface of the pretzel).

$\mathrm{ct}(X)$ versus $\mathrm{cat}(X)$ - the Lusternik-Schnirelman category.

Definition (Lusternik-Schnirelman category, Geometric category)

- $\mathrm{cat}(X) :=$ min. elements of a cover $\mathcal{U} = \{U_i\}$ such that $U_i \rightsquigarrow *$ in X.

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- *geometric category*, defined as the minimal cardinality of a cover of X by open contractible sets.
  The geometric category is not a homotopy invariant of X, so one defines the *strong category*, $\mathrm{Cat}(X)$ as the min of geometric categories of $Y \simeq X$, [11, Proposition 3.15]

Example

$$\mathrm{ct}(S^n) = n+2, \text{ (we show it later)} \qquad \mathrm{cat}(S^n) = 2$$

- the difference arbitrary large.

For the wedge on n circles W_n we have $\mathrm{sct}(W_n) = n+2$, while $\mathrm{ct}(W_n) = \left\lceil \frac{3+\sqrt{1+8n}}{2} \right\rceil$ (see [23, Proposition 4.1])

Estimates by L-S category

Theorem (Govc - Marzantowicz - Pavesič, [17])

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For real and complex projective spaces
$\mathrm{cat}(\mathbb{R}P^n) = \mathrm{cat}(\mathbb{C}P^n) = n+1$

Corollary

$$\mathrm{ct}(\mathbb{R}P^n) \geq \frac{(n+1)(n+2)}{2}$$

$$\mathrm{ct}(\mathbb{C}P^n) \geq \frac{(n+1)(n+2)}{2}$$

We show that the above can be improved.

More fine version of previous theorem

Theorem

$\mathrm{ct}(X) \geq 1 + \mathrm{hdim}(X) + \frac{1}{2}\,\mathsf{cat}(X)(\mathsf{cat}(X) - 1)$, where $\mathrm{hdim}(X)$ is the homotopy dimension. E.g. $\mathrm{ct}(S^d) \geq d + 2$

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$\mathrm{ct}(X) \geq 1 + \mathrm{hdim}(X) + \frac{1}{2}\,\mathrm{cat}(X)(\mathrm{cat}(X) - 1)$, *where* $\mathrm{hdim}(X)$ *is the homotopy dimension. E.g.* $\mathrm{ct}(S^d) \geq d + 2$

A triangulation of a manifold is *combinatorial* if the links of all vertices are triangulated spheres.

Corollary

Let K be a combinatorial triangulation of a d-dimensional and c-connected closed manifold M. Then K has at least

$1 + d + c \cdot (\mathrm{cat}(M) - 2) + \frac{1}{2}\,\mathrm{cat}(M)(\mathrm{cat}(M) - 1)$ *vertices.*

We used the known inequality (see [11])

$$\mathrm{cat}(V) \leq \frac{\mathrm{hdim}(V)}{c+1} + 1 \tag{1}$$

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Let X be a topological space. We define the cuplength of X as

$$\mathrm{clength}(X) := \max_k \; : \; \exists \, x_i \in \tilde{H}^*(X;\mathcal{R}) \, , \; 1 \leq i \leq k$$

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Theorem (Lusternik-Schnirelmann)

$$\mathsf{cat}(X) \; \geq \; \mathrm{clength}\,(X) + 1$$

We have

$$\mathrm{clength}\,\mathbb{R}\mathrm{P}^{\mathrm{n}} = \mathrm{n}, \quad \mathrm{clength}\,\mathbb{C}\mathrm{P}^{\mathrm{n}} = \mathrm{n}$$

For given n-tuple of positive integers $i_1, \ldots, i_n \in \mathbb{N}$ we say that X *admits an essential* $(i_1, \ldots, i_n)$*-product* if there are coh. classes $x_k \in H^{i_k}(X)$, such that $x_1 \cdot x_2 \cdot \ldots \cdot x_n$ is non-trivial.

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If X admits an essen. $(i_1, \ldots, i_n)$-prod. then so does every $Y \simeq X$.

[Definition, [17]]

We define the *covering type of the n-tuple of positive integers* $(i_1, \ldots, i_n)$ as

$$\mathrm{ct}(i_1, \ldots, i_n) := \min \left\{ \mathrm{ct}(X) \mid X \text{ admits an ess. } (i_1, \ldots, i_n)-\mathrm{prod.} \right\}$$

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Proposition ([17])

$$\mathrm{ct}(X) \geq \max\{\mathrm{ct}(|x_1|, \dots, |x_n|) \mid \text{ for all } 0 \neq x_1 \cdots x_n \in H^*(X)\}$$

Lemma ([17])

If X has non-trivial reduced homology groups in different dimensions, then $\mathrm{ct}(X) \geq \mathrm{hdim}(X) + 3$.

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We are ready to prove the main result of this section, an 'arithmetic' estimate for the covering type of a *n*-tuple:

Theorem (Govc-Marzantowicz- Pavešič, [17])

$$\mathrm{ct}(i_1, \ldots i_n) \geq i_1 + 2\, i_2 + \cdots + n i_n + (n+1)$$

If $i_1, \ldots i_n$ *are not all equal, then*

$$\mathrm{ct}(i_1, \ldots i_n) \geq i_1 + 2\, i_2 + \cdots + n i_n + (n+2)$$

Corollary

The covering type of projective spaces is bounded by:
$\mathrm{ct}(\mathbb{R}P^n) \geq \frac{1}{2}(n+1)(n+2)$, $\mathrm{ct}(\mathbb{C}P^n) \geq (n+1)^2$, $\mathrm{ct}(\mathbb{H}P^n) \geq (n+1)(2n+1)$.

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For $\mathbb{R}P^n$ and $\mathbb{C}P^n$ these numbers are equal to the best know estimate obtained by use of the combinatorial methods, so that numerically it reproves the result of [3]. For $\mathbb{H}P^n$ there is not known a construction (estimate) of a "minimal" triangulation.

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Corollary

For a product $X = S^{i_1} \times \cdots \times S^{i_n}$, *where* $i_1 \leq \ldots \leq i_n$ *are not all equal, Thm. 19 yields* $\mathrm{ct}(X) \geq i_1 + 2\, i_2 + \cdots + n i_n + (n+2)$, *while for a product of spheres of the same dimension we get*

$$\mathrm{ct}((S^i)^n) \geq \frac{(n+1)(ni+2)}{2}\,.$$

The last estimate can be sometimes improved by ad-hoc methods

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Corollary

The covering type of unitary groups is estimated as

$$\mathrm{ct}(U(n)) \geq \frac{1}{6}(4n^3 + 3n^2 + 5n + 12) \ \ \textit{and} \ \ \mathrm{ct}(SU(n)) \geq \frac{1}{6}(4n^3 - 3n^2 + 5n + 6).$$

These estimates are new.

$G_k(\mathbb{R}^n)$ all k-dimensional linear subspaces of $\mathbb{R}^n$. A manifold (actually a non-singular alg. variety) of dim $k \cdot (n-k)$. Subspace $\Longleftrightarrow$ orthogonal complement $\Longrightarrow$ a homeomorphism $G_k(\mathbb{R}^n) \simeq G_{n-k}(\mathbb{R}^n)$, so we may that $k \leq n/2$.

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In [42] we used R. Stong's [36] determination of the height of the Stiefel-Whitney class w_1 in $H^*(G_k(\mathbb{R}^n))$, and of non-trivial products in the top dimension of $H^*(G_k(\mathbb{R}^n))$ for $k = 2, 3, 4$.

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The $\mathbb{F}_2$-cohomology ring of $G_k(\mathbb{R}^n)$ was determined by A. Borel (cf. [36]). Let $w_1, \ldots, w_k \in H^*(G_k(\mathbb{R}^n); \mathbb{F}_2)$ denote the Stiefel-Whitney classes of the canonical k-dimensional v. bundle over $G_k(\mathbb{R}^n)$, and let $\overline{w}_1, \ldots, \overline{w}_{n-k} \in H^*(G_k(\mathbb{R}^n); \mathbb{F}_2)$ denote the dual Stiefel-Whitney classes (i.e., the Stiefel-Whitney classes of the orthogonal complement of the canonical bundle).

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The S-W classes and their duals are related by

$$w \cdot \overline{w} = (1 + w_1 + \ldots + w_k) \cdot (1 + \overline{w}_1 + \ldots + \overline{w}_{n-k}) = 1,$$

so the dual classes recursively expressed as polynomials in $w_1, \ldots, w_k$.

Proposition (Govc-Marzantowicz-Pavesič, [42], TMNA 2020)

Let $2^s < n \leq 2^{s+1}$. *Then the following classes in* $H^{4(n-4)}(G_4(\mathbb{R}^n); \mathbb{F}_2)$ *are non trivial* ($0 \leq r < s$, $0 \leq t < 2^r$)*:*

$$
\begin{array}{ll}
\text{if } n = 2^s + 1, & \text{then } w_1^{2^{s+1}-2} \cdot w_2^{2^s-5} \neq 0, \; w_1^{2^{s+1}-1} \cdot w_2^{2^s-7} \cdot w_3 \neq 0; \\
\text{if } n = 2^s + 2^r + 1 + t, & \text{then } w_1^{2^{s+1}-2} \cdot w_2^{2^s+2^{r+1}-5} \cdot w_4^t \neq 0, \\
& \text{also } w_1^{2^{s+1}-1} \cdot w_2^{2^s+2^{r+1}-7} \cdot w_3 \cdot w_4^t \neq 0 \text{ if } r > 0;
\end{array}
$$

Consequences of estimates of $\mathrm{ct}(X)$ by the weighted lengths of X (cf. [17]) presented in Section 8

Theorem ([42], TMNA 2020)

Assume $k \geq 4$, and let $2^s < n \leq 2^{s+1}$ for some s. Then

$$f_0(G_k(\mathbb{R}^n)) \geq 2^s(2^{s+1} + 1).$$

Thu, $f_0(G_k(\mathbb{R}^n))$ increases (at least) as a quadratic funct. of n.

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Proof: By Proposition **??** we have $w_1^{2^{s+1}-1} \neq 0$, then by Prop. **??**

$$f_0(G_k(\mathbb{R}^n)) \geq \frac{2^{s+1}(2^{s+1}+1)}{2} = 2^s(2^{s+1}+1).$$

For the second statement, observe that $n \leq 2^{s+1} \leq 2n-2$, therefore

$$\frac{n(n+1)}{2} \leq 2^s(2^{s+1}+1) \leq (n-1)(2n-1).$$

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[13, CH.1, §7] and [14] that the cohomology ring of a compact Lie group G with coefficients in a field $\mathcal{R}$ of characteristic 0 is of the form

$$H^*(G;\mathcal{R}) = \bigwedge_{\mathcal{R}}[y_{2m_1+1}, y_{2m_2+1}, \ldots, y_{2m_l+1}]$$

where $0 \leq m_1 \leq m_2 \leq \cdots \leq m_l$ and the sequence $(m_1, m_2, \ldots, m_l)$ is called "the rational type" of G. [14]: If G is a simple compact Lie group then

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$$\sum_{j=1}^{l} m_j = \frac{1}{2}(d - l)\,, \qquad (2)$$

where $l = \mathrm{rk}\, G$ is the rank of G (i.e. the dimension of maximal torus $\mathbb{T} \subset G$) and $d = \dim G$ is the dimension of G.

Theorem (Duan-Marzantowicz-Zhao,[41], Topology & Applications 2020)

Let G be a compact simple Lie group and $(m_1, m_2, \ldots, m_l)$ its rational type. Then $(*)$ $\quad \mathrm{ct}(G) \geq l + 1 + \sum_{j=1}^{l} j\,(2m_j + 1).$

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Corollary (DMZ, [41])

Let G be a compact simple Lie group of rank l and dimension d. Let $\theta = \{\frac{d-l}{2l}\}$ be the fractional part of $\frac{d-l}{2l}$. Then
$\mathrm{ct}(G) \geq \frac{(l+1)(d+2)}{2} + (\theta + \theta^2)\, l^2 - 2\theta$. *In particular*
$\mathrm{ct}(G) \geq \frac{(l+1)(d+2)}{2}$.

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The above give straight the estimates of $f_0(G)$ for every triangulation.

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$$\mathrm{ct}(Sp(n)) \geq \frac{1}{6}(8n^3+9n^2+7n+1),$$

$$\mathrm{ct}(SO(n)) \ \geq \ \frac{1}{6}(4n^3+3n^2+5n+12).$$

Thus the same estimates of $f_0(G)$ for any triangulations of each of these groups.

Exceptional Lie groups

For the five simply-connected exceptional Lie groups G, the structure of the algebra $H^*(G;\mathbb{F}_p)$ as a module over the Steenrod algebra A_p, has been determined by Duan and Zhao in [40, Theorem 1.1] As a consequence we got [41]

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Proposition (DMZ, [41])

The covering type of the exceptional Lie groups G_2, F_4, E_6, E_7 and E_8 have the lower bounds $44, 259, 486, 1288$ and 5870, respectively. Consequently $44, 259, 486, 1288$ and 5870 are lower estimates of $f_0(G_2)$, $f_0(F_4)$, $f_0(E_6)$, $f_0(E_7)$, and $f_0(E_8)$ respectively.

In recent works [[42], TMNA 2020], [[41], Top. & App.] we estimated from below the number of all simplices of a triangulation of the spaces for which we are able to estimate the number $f_0(X)$ of the minimal triangulation, i.e. we estimated the size of total minimal triangulation. Our computation has two main ingredients.

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1. Lower bounds for the number of vertices in a triangul. of space whose coh. admits certain non-trivial products given in the previous work [17].

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1. Lower bounds for the number of vertices in a triangul. of space whose coh. admits certain non-trivial products given in the previous work [17].
2. The Lower Bound Theorem (LBT) of Barnette [5], [6], Gromov [18], Kalai [22], and its more sharp versions of Adiprasito [1], and earlier I. Novik and E. Svartz [31], [32], [33] that estimates the number of faces in a triangulation of a (pseudo)manifold with a given number of vertices.

LBT states that the number of i-faces of a triangulation of d manifold are equal to the number of i-faces of the standard simplex of f_0 vertices diminished by a term which does not depend on f_0.

Theorem (LBT)

Let K be a triangulation of a d-dimensional closed manifold. Then

$$f_i(K) \geq f_0(K) \cdot \binom{d+1}{i} - i \cdot \binom{d+2}{i+1} \quad \textit{for } i = 0, \ldots, d-1$$

and

$$f_d \geq f_0 \cdot d - (d+2)(d-1).$$

Moreover, by adding up all inequalities we obtain an estimate for the total number of simplices in K:

$$f_0 + \ldots + f_d \geq 2[(f_0 - d)(2^{d+1} - 1) + 1]$$

A stronger version of the GLBT in which appear the Betti numbers β_j of $|K|$, that reflects the topology of K and improves the estimates

Theorem (GLBT)

Under the previous assumptions, we have the following bounds:

$$f_i \geq f_0 \cdot \binom{d+1}{i} - i \cdot \binom{d+2}{i+1} + \binom{d+1}{i+1} \sum_{j=0}^{i} \binom{i}{j} \beta_j$$

$$+ \sum_{j=2}^{\lfloor \frac{d+2}{2} \rfloor} \left[\binom{d+2-j}{d+1-i} - \binom{j}{d+1-i} \right] \binom{d+1}{j-1} \beta_{j-1} \quad \textit{for } i = 0, \ldots, d-1$$

and

$$f_d \geq f_0 \cdot d - (d+2)(d-1) + \sum_{j=0}^{d-1} \binom{d}{j} \beta_j + \sum_{j=2}^{\lfloor \frac{d+2}{2} \rfloor} (d+2-2j) \binom{d+1}{j-1} \beta_{j-1}.$$

Theorem (GMP, [42])

As above, let f_d be the number of facets in a minimal triangulation of $G_k(\mathbb{R}^{n+k})$. Then we have:

$$f_d \geq 2^{nk} + \frac{k}{2} \cdot (n^3 + n^2) + \frac{k(k+2)(k-1)}{2} \cdot n.$$

In particular, it grows as an exponential function in n.

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In particular, it grows as an exponential function in n.

Remark

Using similar estimates, one can also show that the minimal number of faces f_{d-j} in $G_k(\mathbb{R}^{n+k})$ for a fixed k and a fixed codimension j must grow exponentially as $n \to \infty$.

Surprising: Grassmannians admit simple decompositions into the Schubert cells. The standard of $G_k(\mathbb{R}^n)$ has $\binom{n}{k}$ cells (only one 0-dimensional and one top-dimensional cell). A contrary

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Example (The number of simplices in any triangulation is huge:)

$G_3(\mathbb{R}^9)$ is 18-dimensional and every triangulation requires at least 185 vertices. As a consequence, every triangulation of $G_3(\mathbb{R}^9)$ must have at least

$$185 \cdot 18 - (18 + 2) \cdot (18 - 1) = 2990$$

facets and at least

$$2((185 - 18) \cdot (2^{19} - 1) + 1) > 175 \cdot 10^6$$

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$G_4(\mathbb{R}^9)$ is 20-dimensional and $f_0(G_4(\mathbb{R}^9)) \geq 242$. Therefore, every triangulation of $G_4(\mathbb{R}^9)$ requires more than 4422 facets and more than $930 \cdot 10^6$ simplices.

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$G_4(\mathbb{R}^9)$ is 20-dimensional and $f_0(G_4(\mathbb{R}^9)) \geq 242$. Therefore, every triangulation of $G_4(\mathbb{R}^9)$ requires more than 4422 facets and more than $930 \cdot 10^6$ simplices. The number of 4-dimensional simplices, whose links should be examined to compute the first rational Pontrjagin class by means of Gaifulin's formula of [15] exceeds 1.3 million.

Theorem

For the classical compact Lie groups $U(n)$, $SU(n)$, $SO(n)$, and $Sp(n)$ we have the following estimates of number of facets, i.e. simplices of highest dimension $d = \dim G$:

1) $f_d(U(n)) \geq \frac{1}{6}(4n^5 - 3n^4 + 5n^3 + 6n^2 + 12) + 2^n - 1,$
2) $f_d(SU(n)) \geq \frac{1}{6}(4n^5 - 9n^4 + n^3 + 15n^2 + 6) + 2^{n-1} - 1,$
3) $f_d(SO(2n+1)) \geq$
   $\frac{1}{3}(32n^5 + 64n^4 + 64k^3 + 38k^2 + 9k + 6) + 2^n - 1\,,$
4) $f_d(SO(2n)) \geq \frac{1}{3}(32n^5 - 16n^4 + 16n^3 - 2k^2 - 3k + 6) + 2^n - 1,$
5) $f_d(Sp(n)) \geq \frac{1}{6}(8n^5 + 15n^4 + 12n^3 + 11n^2 + 6n + 12) + 2^n - 1\,.$

Consequently, in all these cases the number f_d of facets grows exponentially in n, thus in $l = \mathrm{rank}\, G$ and $d = \dim G$.

Our estimations of number of all simplices, i.e. of all dimensions, which are necessary to triangulations of the exceptional Lie groups are given as follows. The underlined is the maximal (best) estimate for each group.

G	$\mathbb{F}_2$	$\mathbb{F}_3$	$\mathbb{F}_5$	$\mathbb{Q}$
G_2	$\underline{11746824}$	4059144	4059144	4059144
F_4	1.57775×10^{25}	$\underline{3.50157 \times 10^{25}}$	9.63191×10^{24}	9.63191×10^{24}
E_6	1.66706×10^{38}	$\underline{2.57662 \times 10^{38}}$	8.22191×10^{37}	8.22191×10^{37}
E_7	1.44756×10^{65}	$\underline{1.23839 \times 10^{73}}$	1.68159×10^{64}	1.68159×10^{64}
E_8	$\underline{1.85929 \times 10^{121}}$	1.30821×10^{120}	7.2883×10^{119}	1.40822×10^{119}

Notify, that the estimates depend on the coefficients of cohomology we use to the analysis.

At the end , we would like to point out very recent work of K. Knudson [43]. He defined the notion of an approximate triangulation for a manifold M embedded in Euclidean space.

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The basic idea is to build a nested family of simplicial complexes whose vertices lie in M and use persistent homology to find a complex in the family whose homology agrees with that of M.

The author gave samples and calculations provided with computer program implementations. The test spaces are lower dimensional Grassmann manifolds and a verification is made by the comparison with estimates of sizes of triangulations derived in [42]

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Beautiful friend
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My only friend, **the end**