INTEGRATION THE LOADED KORTEWEG-DE VRIES EQUATION IN THE CLASS OF STEPLIKE FUNCTION

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INTRODUCTION.

It is known, that the Korteweg-de Vries equation can be integrated with Inverse Scattering Method [1]. In the works [2,3], the Korteweg-de Vries equations with a self-consistent source were integrated for a class of initial data of “step” type; in particular, laws of evolution of the scattering data were established. In applications of the method of inverse scattering transformation one looks for pairs of operators B and L such that the equation has some interesting nonlinear evolution equation for functions $u(x,t)$ that occur as potentials in the operator L. For the successful application of the method two further ingredients are needed: 1. the inverse scattering problem must be solved so that the potentials $u(x,t)$ can be reconstructed from scattering data; 2. and that one must be able to determine the evolution of the scattering data with t.

In this work, we will consider the loaded Korteweg-de Vries equation

$$u_t - 6uu_x + u_{xxx} + \gamma(t)u(0,t)u_x = 0, \tag{1}$$

where $u = u(x,t)$, $x \in R$, $t \geq 0$, $\gamma(t)$ – is an arbitrary, continuous function. The function $u = u(x,t)$ is a sufficiently smooth and tending to its limits steplike $(c > 0)$

$$\int_{-\infty}^{0}(1-x)\left|u(x,t)\right|dx + \int_{0}^{\infty}(1+x)\left|u(x,t)-c^2\right|dx + \sum_{k=1}^{3}\int_{-\infty}^{\infty}\left|\frac{\partial^k u(x,t)}{\partial x^k}\right|dx < \infty \tag{2}$$

The equation (1) is considered with initial condition

$$u\big|_{t=0} = u_0(x)\,,\ x \in R^1, \tag{3}$$

where $u_0(x)$ function satisfies the conditions $(c > 0)$:

1. $\int\limits_{-\infty}^{0}(1-x)\left|u_0(x)\right|dx < \infty,\qquad \int\limits_{0}^{\infty}(1+x)\left|u_0(x)-c^2\right|dx < \infty,$

2. Suppose that, the equation $-y'' + u_0(x)y = \lambda y,\ x \in R^1$ has $\lambda_1(0), \lambda_2(0), \ldots, \lambda_N(0)$ negative eigenvalues.

In this work the solution $u(x,t)$ of the loaded Korteweg-de Vries equation (1) in the class of steplike function (2) with initial condition (3) is obtained via Inverse Scattering Method.

SCATTERING PROBLEM

Consider the Sturm–Liouville equation on the line $x \in R^1$

$$Ly \equiv -y'' + u(x)y = k^2 y\,, \tag{4}$$

where the potential $u(x)$ is real, locally summable, and has different limits ar infinities of different signs:

$$\lim_{x\to-\infty} u(x) = 0,\quad \lim_{x\to\infty} u(x) = c^2,\ \ c \ge 0. \tag{5}$$

We assume that $u(x)$ tends to its limits fast enough, so that

$$\int\limits_{-\infty}^{0}(1-x)\left|u(x)\right|dx < \infty,\qquad \int\limits_{0}^{\infty}(1+x)\left|u(x)-c^2\right|dx < \infty. \tag{6}$$

The scattering problem for Eq. (4) under condition (6) was considered in the works [2-4].

We set $l = \sqrt{k^2 - c^2}$ we choose a branch of the square root in the plane with the cut $[-c,c]$ such that $\operatorname{Im} l > 0$ for $\operatorname{Im} k > 0$ and $\operatorname{sign} l = \operatorname{sign} k$ for real k and $|k| > c$, respectively.

Denote by $f^+(x,k)$ and $f^-(x,k)$ respectively, the Yost solutions of Eq. (4) with the following asymptotics:

$$\lim_{x\to\infty} f^+(x,k)\exp(-ilx)=1, \quad (\operatorname{Im} l \geq 0), and \quad \lim_{x\to-\infty} f^-(x,k)\exp(ikx)=1, \quad (\operatorname{Im} k \geq 0). \tag{7}$$

Under conditions (6), such solutions exist, are unique, and are regular functions of l and k for $\operatorname{Im} l > 0$ and $\operatorname{Im} k > 0$, respectively. They can be represented in terms of the transformation operators as follows:

$$f^+(x,k) = \exp(ilx) + \int_x^\infty A_1(x,z)\exp(ilz)dz, \qquad f^-(x,k) = \exp(-ikx) + \int_{-\infty}^x A_2(x,z)\exp(-ikz)dz. \tag{8}$$

In representation (8), the kernels $A_1(x,z)$ and $A_2(x,z)$ are real-valued functions which are related to the potential $u(x)$ by the equalities

$$u(x) = 2\frac{dA_2(x,x)}{dx}, \qquad u(x) = c^2 - 2\frac{dA_1(x,x)}{dx}. \tag{9}$$

Under condition (6), Eq. (4) has solutions $\psi_1(x,k)$ and $\psi_2(x,k)$ with the following asymptotics:

$$\psi_1(x,k) \sim \begin{cases} e^{ikx} + S_{21}(k)e^{-ikx}, & x \to -\infty, \\ S_{22}(k)e^{ilx}, & x \to \infty, \end{cases} \tag{10}$$

$$\psi_2(x,k) \sim \begin{cases} S_{11}(k)e^{-ikx}, & x \to -\infty, \\ e^{-ilx} + S_{12}(k)e^{ilx}, & x \to \infty \quad (|k| > c). \end{cases} \tag{11}$$

The matrix $S_{ij}(k),\ i,j = 1,2$ is called the S-matrix of Eq. (4). The coefficients $S_{ij}(k)$ are continuous in $k\,(-\infty < k < \infty)$. The coefficients $S_{11}(k)$ and $S_{22}(k)$ are limit values of functions that are meromorphic in the upper half-plane and have simple poles at the points $k_n = i\chi_n\ (\chi_n > 0),\ n = 1,2,3,\ldots,N$, where $\lambda_n = -\chi_n^2$ are eigenvalues of the operator L and $f^-(x,i\chi_n) = B_n f^+(x,i\chi_n)$. Note that $S_{22}(k) = 0$.

It is possible to recover the potential $u(x)$ of Eq. (4) from its S-matrix. Set

$$m_n^+ = \left(\int_{-\infty}^{\infty} f^+(x,i\chi_n)^2\,dx \right)^{-1}, \; n = 1,2,\ldots,N,$$

The kernel $A_1(x,y)$ in (8) satisfy the integral Gelfand–Levitan–Marchenko equations:

$$A_1(x,y)+\Omega_1(x+y)+\int_x^\infty \Omega_1(z+y)A_1(x,z)dz=0 \ \ (y>x),$$

where

$$\Omega_1(x)=\sum_{n=1}^N m_n^+ e^{-\sqrt{\chi_n^2+c^2}x}+\frac{1}{2\pi}\int_{-\infty}^\infty S_{12}(k)e^{ilx}dl+\frac{1}{2\pi}\int_0^c |S_{22}(k)|^2 e^{-\sqrt{c^2-k^2}x}dk.$$

Now the potential $u(x)$ is determined by formulas (9).

The S-matrix and the set $\{\chi_1,\chi_2,\dots,\chi_N, m_1^+, m_2^+,\dots,m_N^+\}$ are called the scattering date of Eq. (4).

Following theorem is valid

Theorem. If the potential $u(x,t)$ is a solution the problem (1)-(3) then the scattering data of the Eq. (4) with the function $u(x,t)$ depend on t as follows:

$$\frac{dS_{21}}{dt}=-(8ik^3-2ik\gamma(t)u(0,t))S_{21}, \quad (\operatorname{Im}k=0) \quad \text{for } |k|>c$$

$$\frac{dS_{22}}{dt}=(-4ik^3+(2c^2+4k^2)il-i\gamma(t)u(0,t)(l-k))S_{22}, \quad \text{for } |k|\le c$$

$$\frac{dS_{22}}{dt}=\left(-4ik^3+(2c^2+4k^2)\sqrt{c^2-k^2}+\frac{\sqrt{c^2-k^2}}{4\pi}J_1-\frac{ik}{2}\gamma(t)u(0,t)\left(1+|S_{21}|^2\right)dx+\frac{\sqrt{c^2-k^2}}{2\pi}\operatorname{Im}J_2\right)S_{22},$$

and

$$\frac{d\chi_n}{dt}=0, \quad n=1,2,3,\dots,N,$$

$$\frac{dB_n}{dt}=\left(-4\chi_n-(2c^2+4k^2)\sqrt{c^2-k^2}+\gamma(t)u(0,t)\left(\sqrt{\chi_n^2+c^2}+\chi_n\right)\right)B_n, \quad n=1,2,3,\dots,N,$$

where

$$J_1(t) = \text{v.p.}\int_{-c}^{c} -\frac{2k'^2(1+|S_{21}|)^2\gamma(t)u(0,t)}{k'\sqrt{c^2-k'^2}\,(k'-k)}dk' ,$$

$$J_2(t) = \left(\int_{-\infty}^{-c} + \int_{c}^{\infty}\right) -\frac{2l\gamma(t)u(0,t)(l-k')}{(k'^2-c^2)(k'-k)}dk' .$$

The taken relations determine completely the evolution of the scattering data for the Eq. (4) which allows as to find the solution of problem for (1-3) by using Inverse scattering problem method.

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