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Having Fun with Designs

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8ECM MiniSymposium

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Acknowledgement
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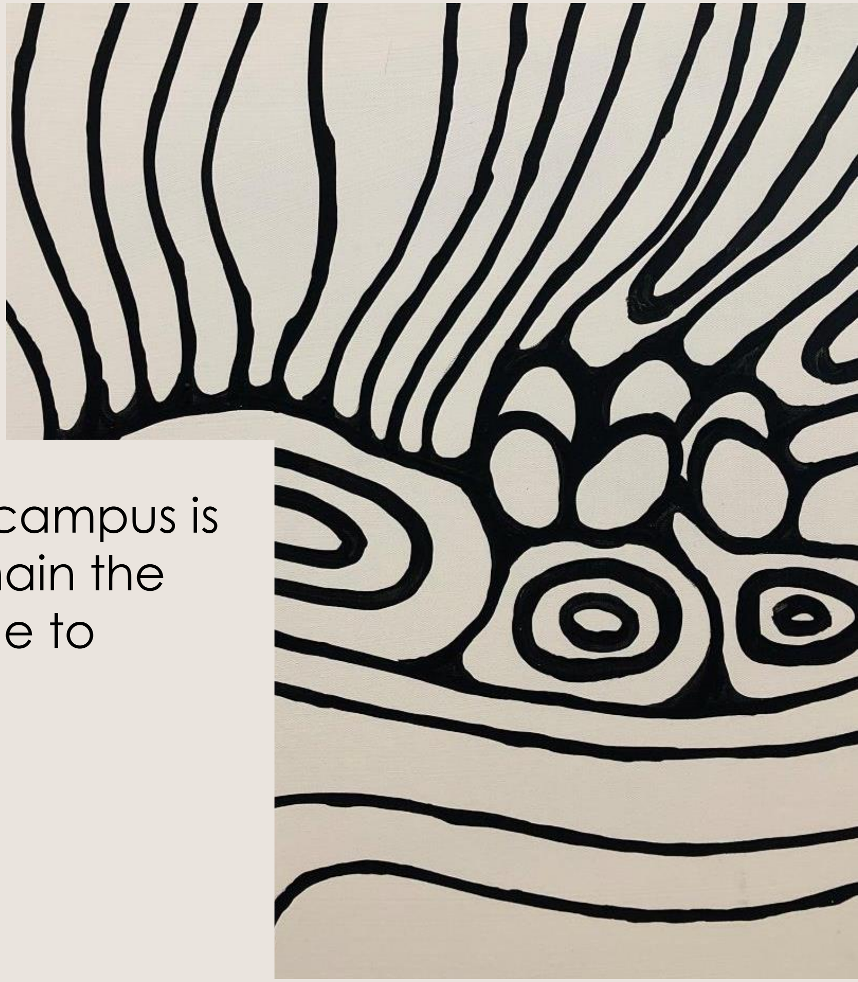


The University of Western Australia acknowledges that its campus is situated on Noongar land, and that Noongar people remain the spiritual and cultural custodians of their land, and continue to practise their values, languages, beliefs and knowledge.

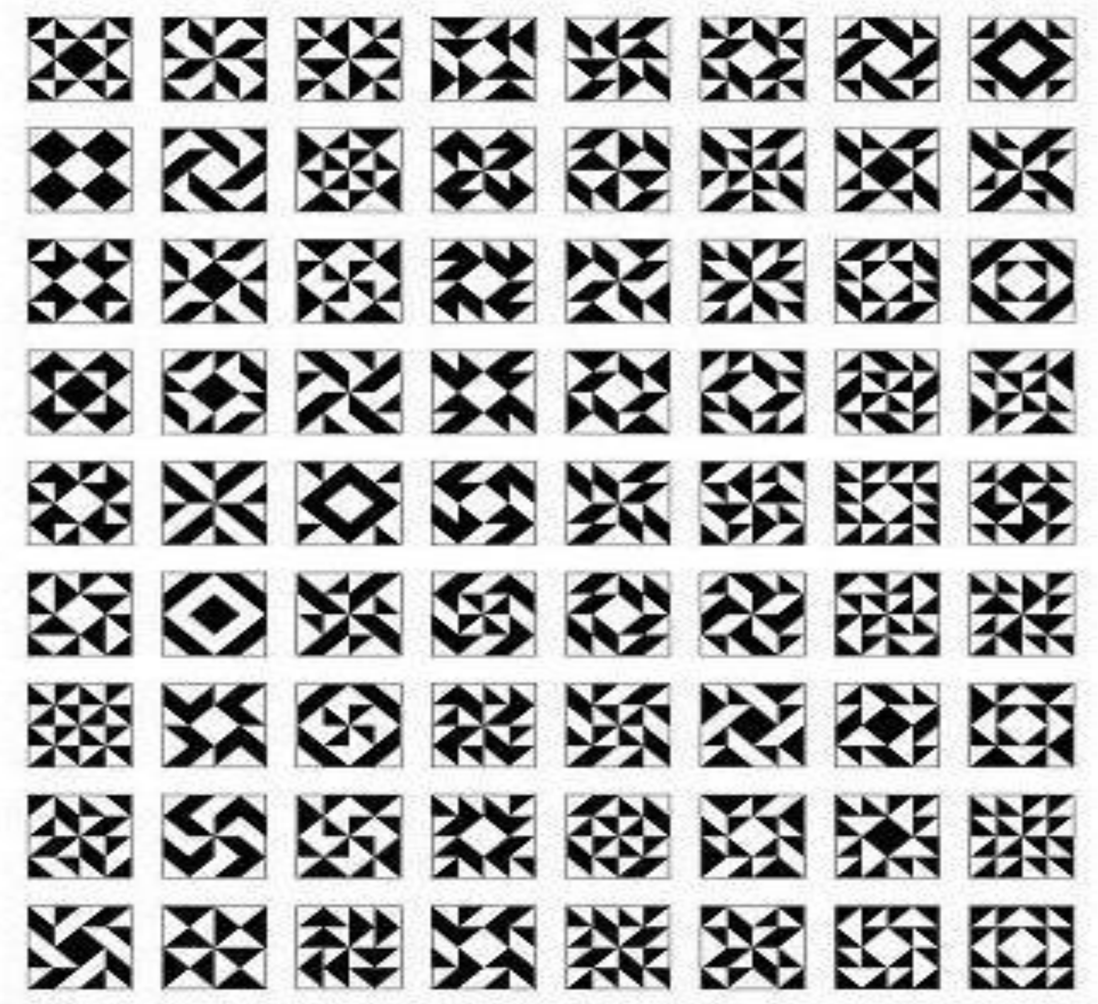
Artist: Dr Richard Barry Walley OAM

Main theme

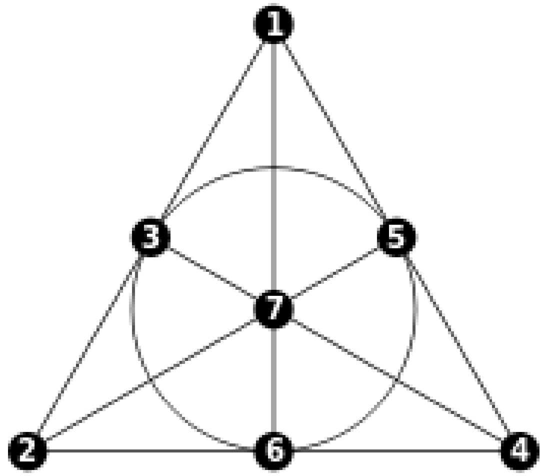
Imprimitive Symmetry

2-designs

- Delandtsheer-Doyen parameters for block transitivity (1989)
- Davies' bound for flag-transitivity (1987)
- Our new results about each: collaboration with Alice Devillers and Carmen Amarra



t-designs



1 t-(v,k, λ) design D = (P, B) P set of v points B set of blocks
Block is a k-subset of P
Each t-subset of points contained in λ blocks

2 Symmetry: Aut(D) = { permutations of P mapping blocks to blocks }
If $t \geq 2$ then Block-transitive ⟹ point-transitive

Want $t \geq 2$ and
$G \leq Aut(D)$ block-transitive

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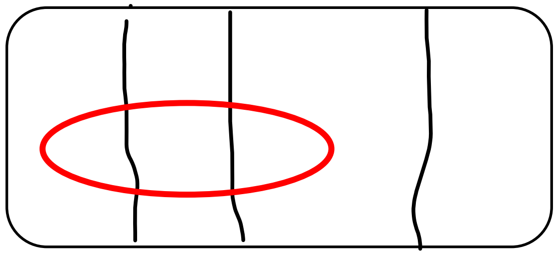
2-design imprimitive symmetry

1 2-(v,k, λ) design D = (P, B) $G \leq Aut(D)$ block-transitive

2 G point-imprimitive $\exists$ G − invariant point − partition Σ $v = c \cdot d$

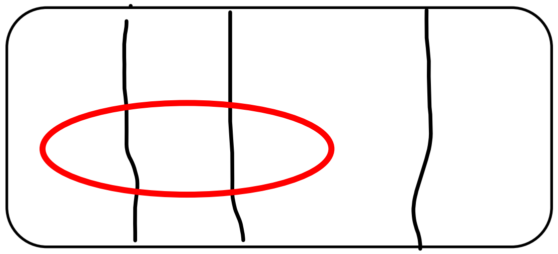


G preserves both the block set B and the partition Σ
How do they interact?

Delandtsheer-Doyen Theorem

2-(v,k, λ) design D = (P, B) $G \leq Aut(D)$ block-transitive, point-imprimitive

- Each block contains n inner pairs (same part of Σ)
- and mc outer pairs (different parts of Σ)
- Set $x = k(k-1)/2$ Then part size $c = \frac{x-n}{m}$ and number of parts $d = \frac{x-m}{n}$



$v = c \cdot d$

(n, m) called the Delandtsheer-Doyen parameters

Bounds on Delandtsheer-Doyen parameters (n,m)

2-(v,k, λ) design   D = (P, B) $G \leq Aut(D)$ block-transitive, preserves point-partition Σ

- G induces transitive group $K \leq Sym(\Sigma)$ on the d parts of Σ , and for $\Delta \in \Sigma$
- Stabiliser G_Δ induces transitive group $H \leq Sym(\Delta)$ on c points of Δ

Amarra, Devillers, P: PairRank(K) = # K-orbits on unordered pairs in Σ $\leq m$

- and PairRank(H) = # H-orbits on unordered pairs in Δ $\leq n$

Can bounds on the Delandtsheer-Doyen parameters be attained?

Cameron-Praeger designs 1993
Pattern of block-part intersections

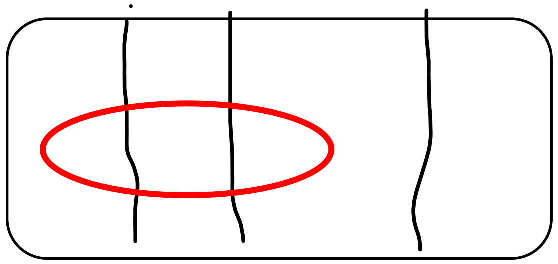
$x_1, x_2, \ldots, x_d$

$\Sigma_i x_i = k$

2-(v, k, λ) design D = (P, B) $G \leq Aut(D)$ block-transitive, point-imprimitive

- Observed: If there exists 2-design with intersections $x_1, x_2, \ldots, x_d$ then exists one with $K = Sym(\Sigma)$ and $H = Sym(\Delta)$
- Here PairRank(K) = PairRank(H) = 1.
- Found all 2-designs with $m = n = 1$ provided $k > 5, k \neq 8$

$v = c \cdot d$



When can the PairRank bounds be achieved with max {m, n} ≥ 2 ?

Some examples meeting the DD-PairRank Bounds

D = Projective Plane PG(2,q). With $v = 1 + q + q^2 = cd$ and $c > 1, d > 1$, and $G = C_v$ a Singer cycle

Here $k = q + 1$, and PairRank $(C_d) = \frac{d-1}{2} = m$, and PairRank $(C_c) = \frac{c-1}{2} = n$

Amarra, Devillers, P: PairRank(K) = # K-orbits on unordered pairs in Σ is $\leq m$

- and PairRank(H) = # H-orbits on unordered pairs in Δ is $\leq n$

Good: m, n can be arbitrarily large
Limitations: exactly which (m,n) arise?
Groups $H = C_c$ and $K = C_d$ both regular

Can bounds be attained with other kinds of groups?

New construction meeting the DD-PairRank Bounds (Amarra, Devillers, P)

Start with $n \geq 2$. Take $\Delta = F_c$ and $c = 1 + 2na$ for some a,

$H = [c].[2a]$, (translations and multiplications by n^{th} powers]

Then PairRank $(H) = n$

To construct a 2-design we need $c + n = \frac{k(k-1)}{2}$ [a triangular number]

And take $d = 1 + \frac{c-1}{n}$, $\mathrm{K} = \mathrm{S_d}$ so PairRank $(K) = m = 1$

"Cleverly" define one block and take all images under $H\ wr\ K$:

Block: n parts of Σ contain two points (different H-orbits on pairs);

$k - 2n$ more parts contain one point

Conditions give a 2-design meeting PairRank bounds with m=1. Which values of n work, and for how many prime powers c?

New construction: when does it work?

Start with $n \geq 2$. So we want prime power $c = 1 + 2na$ for some a, and

we need $c + n = \frac{k(k-1)}{2}$ [a triangular number] with $k \geq 2n$

Write $k = 4nt + r$ with $1 \leq r < 4n$ get a quadratic equation:

$$f_{n,r}(t) = 8n^2t^2 + 2n(2r-1)t + \left(\frac{r(r-1)}{2} - n\right)$$

Whenever $f_{n,r}(t) = c$ is a prime power, get 2-design with $k = 4nt + r$ and $d = \frac{c-1}{n} + 1$, with DD-parameters $(n, 1)$ and *achieving the PairRank bounds*.

- Some n give no solutions, e.g. n = 6, 10, 15
- Most n seem to give many solutions!
- Possibly infinitely many – depends on Bunyakovsky's conjecture in number theory

Bunyakovsky's conjecture - 1857

Our conditions: $n \geq 2$, $1 \leq r < 4n$:

$$f_{n,r}(t) = 8n^2t^2 + 2n(2r-1)t + \left(\frac{r(r-1)}{2} - n\right)$$

Bunyakovsky's Conjecture (for these polynomials):

if $f_{n,r}(t)$ irreducible over integers, and $f_{n,r}(t)$ not identically zero modulo any prime

Then $f_{n,r}(t) = c$ is a prime for infinitely many values of t

First version on arXiv September 2020

- Attracted attention of Gareth Jones and Sasha Zvonkin
- Intensive email exchanges
- Extremely fruitful –e.g. for $n = 2$, they found 12,357,532 prime values for $f_{2,3}(t)$ with $t \leq 10^8$

Bunyakovsky's conjecture - 1857

1857 Bunyakovsky's Conjecture: unfortunately still open

1962 Bateman and Horn: proposed approximation $E(x)$ for which the number of integers $t \leq x$ with $f_{n,r}(t)$ prime

2020 W. Li: improved (more easily computable) version of Bateman-Horn estimate.

2021 Jones & Zvonkin:

- exhaustive search for several (n, r) where we had found multiple examples
- Showed that the numbers of prime values of $f_{n,r}(t)$ for $t \leq 10^8$ were extremely close to the Bateman-Horn estimate $E(10^8)$ in all cases
- E.g. for (n, r)= $(2,3)$ number of prime values up to 10^8 is

  12,357,532 while $E(10^8) = 12,362,961.06$.
- Great for the design construction – also evidence for truth of Bunyakovsky Conjecture

A much shorter second half: flag-transitive, point-imprimitive 2-designs

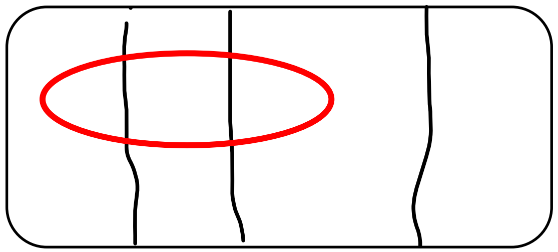
2-(v,k, λ) design   D = (P, B) $G \leq Aut(D)$ flag-transitive, point-imprimitive

- Flag: incident point-block pair – each non-empty part-block intersection must have same size.

1987 Hugh Davies: k and v both bounded above in terms of λ

- **But Hugh gave no explicit upper bounds as functions of λ**

$v = c \cdot d$



Alice Devillers and I set out to find some explicit upper bounds.

Hunting for explicit upper bonds: flag-transitive, point-imprimitive 2-designs

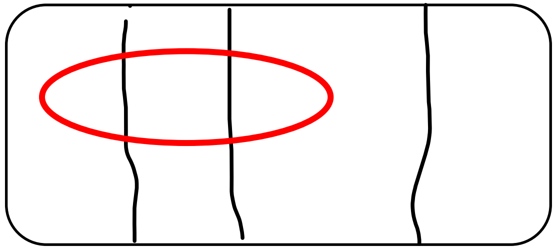
2-(v,k, λ) design D = (P, B) $G \leq Aut(D)$ flag-transitive, point-imprimitive

1993 Cameron, P: Showed that $v \leq (k-2)^2$ and that smallest k is 6

2018 Zhan, Zhou: exactly 14 examples with $k = 6$ $- \textit{all have } v = (k-2)^2 = 16$

- **So just need to find upper bound for k in terms of λ**

$v = c \cdot d$



1961 Higman – McLaughlin: $\lambda \geq 2$

2021 Devillers, Liang, P, Xu: exactly two examples with $\lambda = 2$; both have $k = 6,\ v = 16$

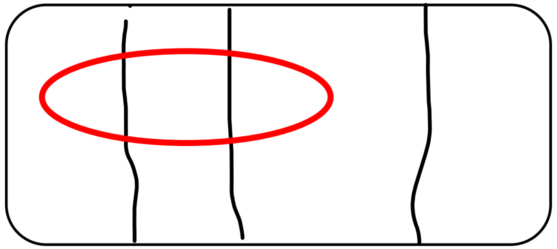
Hunting for explicit upper bonds: flag-transitive, point-imprimitive 2-designs

2-(v,k, λ) design   D = (P, B) $G \leq Aut(D)$ flag-transitive, point-imprimitive

1993 Cameron, P: Showed that $v \leq (k-2)^2$ and that smallest k is 6

2021 Devillers, P: $k \leq 2\lambda^2(\lambda - 1)$ **and so** $v \leq (2\lambda^2(\lambda-1)-2)^2$

$v = c \cdot d$



How good are these bounds?
Not tight for $\lambda = 2$ since bound is
$k \leq 8$ while only examples have $k = 6$

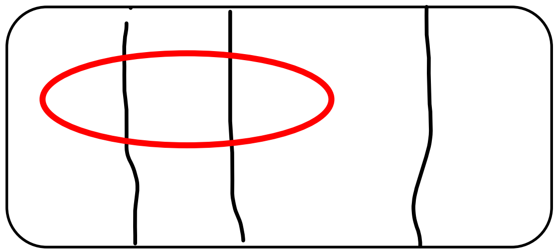
Hunting for explicit upper bonds: flag-transitive, point-imprimitive 2-designs

2-(v,k, λ) design D = (P, B) $G \leq Aut(D)$ flag-transitive, point-imprimitive

2021 Devillers, P: $k \leq 2\lambda^2(\lambda - 1)$ **so** $v \leq (2\lambda^2(\lambda-1)-2)^2$

Analysed $v < 100$ **with** $\lambda \leq 4$ **: exactly eleven examples**

$v = c \cdot d$



Our future objective: find all the examples with $\lambda = 3, 4$

Remaining challenges:

1. Find all Delandtsheer-Doyen parameters (n, m) where there exist block-transitive point-imprimitive 2-designs

   admitting groups H, K with PairRank(H) $= n$ and PairRank(K) $= m$

2. Improve the upper bound $k \leq 2\lambda^2(\lambda - 1)$ for flag-transitive point-imprimitive 2-designs.

3. Prove Bunyakovsky's Conjecture!

Some references

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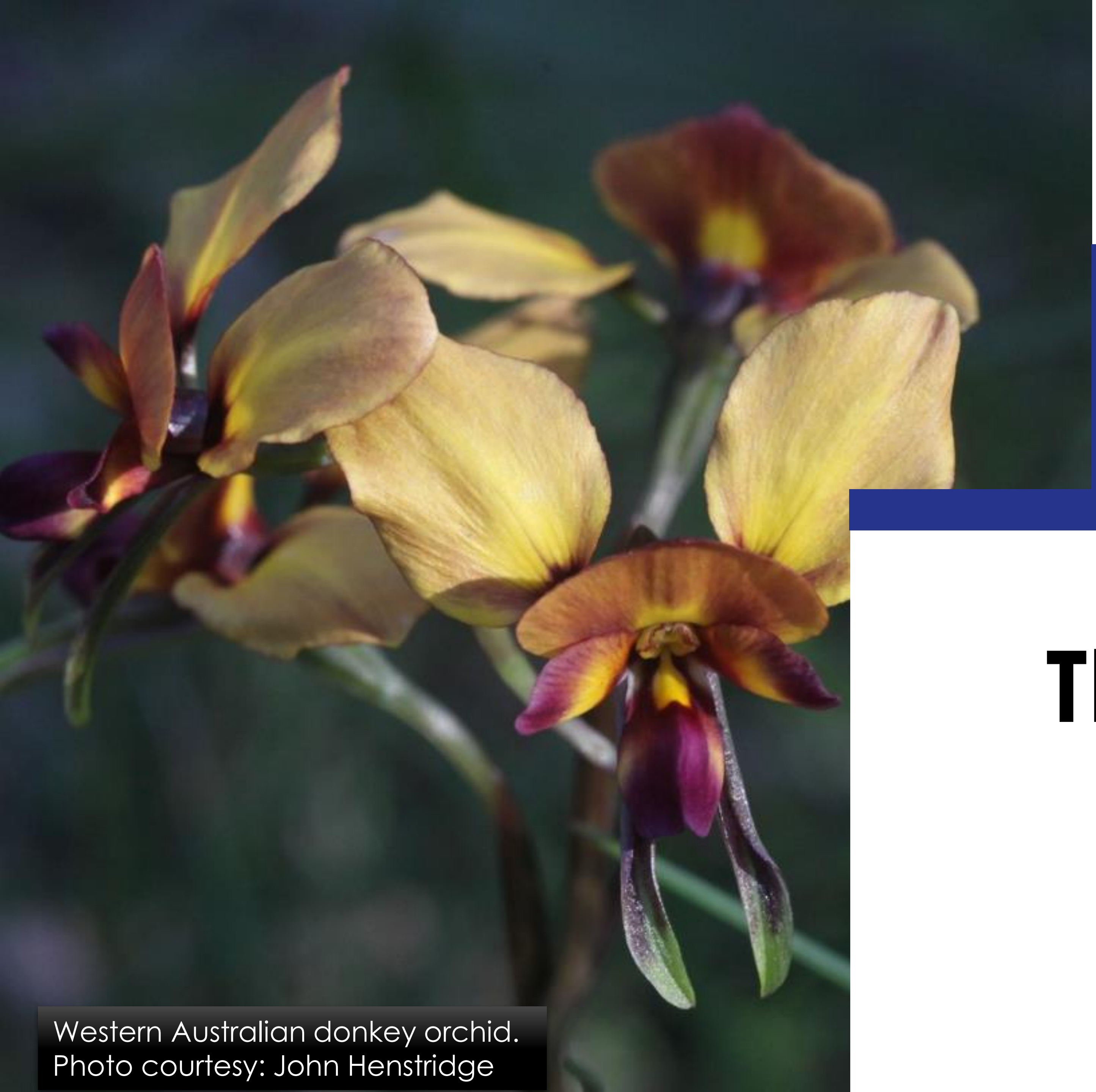
1 A. Delandtsheer and J. Doyen, Most block-transitive t-designs are point-primitive. Geom. Dedicata 29 (1989), 307-310.

2 H. Davies, Flag-transitivity and primitivity, Discrete Math. 63 (1987), 91-93.

3 G. A. Jones and A. K. Zvonkin, Block designs and prime values of polynomials, Available at arxiv:2105.03915v2.

3 C. Amarra, A. Devillers and C. E. Praeger, Delandsheer-Doyen parameters for block-transitive point-imprimitive block designs, Available at arXiv:2009.00282.

4 Devillers and C. E. Praeger, On flag-transitive imprimitive 2-designs, J Combin. Designs 2021, doi:10.1002/jcd.21784



Western Australian donkey orchid.
Photo courtesy: John Henstridge

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Thank you and
Stay safe