Limit states of multi-component discrete dynamical systems

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History

The classical approaches to the conflict problem are known at first due to: the Malthus - Verhulst **population equation** for the internal competition,

$$\frac{\mathrm{d}P}{\mathrm{d}t} = (b-d)P - cP^2,$$

the **logistic equation**,

$$x_{n+1} = r x_n \left(1 - x_n\right),$$

the **Lotka–Volterra equations** for hostile populations,

$$\dot{N} = aN - bNP, \quad \dot{P} = -cP + dNP,$$

the **predator–prey model**, and a large number of collision situations from the game theory, "Theory of Games and Economic Behavior", John von Neumann and Oskar Morgenstern, (1944).

Introduction

We consider an approach to building models with competing parties, which was founded by prof. V.D. Koshmanenko in 2003 [1].

In this approach, analogs of scalar quantities P, N of the Lotka–Volterra equations are changed to vectors $\mathbf{p}$ and $\mathbf{r}$:

$$\mathbf{p} = (p_1, \ldots, p_n), \quad \mathbf{r} = (r_1, \ldots, r_n), \quad n > 1.$$

Coordinates p_i, r_i have a statistical meaning and correspond to the probabilities of finding each of the opponents in the $i-$th region (position) of the resource space $\Omega = \bigcup_{i=1}^{n} \Omega_i$.

In an even more general approach, instead of the vectors $\mathbf{p}$, $\mathbf{r}$, random distributions are used, given by the probability measures μ, ν.

The law of dynamics is written in the form of a system of fairly simple but nonlinear difference or differential equations.

Introduction

It is impossible to understand the global picture of universal conflict phenomena without coming from classic approaches to setting of the problem in terms of the modern theory of dynamical systems and functional analysis.

The important feature of such approach is the coming to statistical (probabilistic) interpretation of results.

All prediction results of conflict interactions will be presented in a form of probabilistic distributions.

One of a new objects is a notion of the conflict transformation, which corresponds to the conflict interaction between two or more alternative sides (opponents).

Let $\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}, n > 1$ be some finite space and a discrete topology is given on Ω. Consider the set of discrete probability measures $\mu_i \in \mathcal{M}_1^+(\Omega)$, $i = \overline{1, m}$ on Ω $(i \geq 1)$. The measure μ_i corresponds to one of the m alternative sides.

Each of these measures μ_i can be identified with a stochastic vector $\mathbf{p}_i = (p_{ij})_{j=1}^n$, where

$$p_{ij} = \mu_i(\omega_j), \quad i = \overline{1, m}, \quad j = \overline{1, n}. \tag{1}$$

The formula of conflict evolutions

Each of the vector $\mathbf{p}_i = (p_{ij})_{j=1}^n$ is associated with vector $\mathbf{p}_i^1 = (p_{ij}^1)_{j=1}^n$ according to the rule defined in terms of coordinates

$$p_{ij}^1 = \frac{1}{z}\big(p_{ij}(\theta + 1) + \tau_j\big), \tag{2}$$

where $\theta = \theta\big(\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_m\big)$ is a finite positive function, $\mathcal{T} = \big(\tau_j\big)_{j=1}^n$ is a vector with non-negative coordinates (attractor index), and z is normalizing denominator.

The formula of conflict evolutions

The mapping $\divideontimes$ generates multi-component discrete dynamical systems with trajectories

$$\{\mathbf{p}_1^t, \mathbf{p}_2^t, \dots, \mathbf{p}_m^t\} \xrightarrow{\divideontimes, t} \{\mathbf{p}_1^{t+1}, \mathbf{p}_2^{t+1}, \dots, \mathbf{p}_m^{t+1}\}, \quad \mathbf{p}_i^{t=0} = \mathbf{p}_i, \quad t = 0, 1, \dots, \tag{3}$$

where the coordinates of each vector $\mathbf{p}_i^t = (p_{ij}^t)_{j=1}^n$ are changed according to the equations

$$p_{ij}^{t+1} = \frac{1}{z^t}\left(p_{ij}^t\left(\theta^t + 1\right) + \tau_j^t\right), \quad t = 0, 1, \dots, \tag{4}$$

$$\theta^t = \theta\left(\mathbf{p}_1^t, \mathbf{p}_2^t, \dots, \mathbf{p}_m^t\right), \quad z^t = 1 + \theta^t + W^t, \quad W^t = \sum_{j=1}^n \tau_j^t > 0, \tag{5}$$

$$\mathbf{w}^t = (w_j^t)_{j=1}^n, \quad w_j^t = \frac{\tau_j^t}{W^t} \tag{6}$$

The limiting states

Theorem 1.

Let all coordinates of stochastic vector $\mathbf{w}^t$ *be monotonic. Then every trajectory of the dynamical system* (3) *with initial state* $\{\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_m\}$ *converges to a fixed point* $\{\mathbf{p}_1^\infty, \mathbf{p}_2^\infty, \dots, \mathbf{p}_m^\infty\}$

$$\mathbf{p}_i^\infty = \lim_{t\to\infty} \mathbf{p}_i^t, \ \forall\ i = \overline{1,m}.$$

Moreover, all limit vectors $\mathbf{p}_i^\infty$ *coincide with the vector*

$$\mathbf{w}^\infty = \lim_{t\to\infty} w^t,$$

that is

$$p_{ij}^\infty = \frac{\tau_j^\infty}{W^\infty} \quad \forall\ i = \overline{1,m}, \quad j = \overline{1,n}. \tag{7}$$

Let us denote the coordinates τ_j^t in one of the following ways:
1) as the minimum value of the j coordinates of the vectors $\mathbf{p}_i^t$

$$\tau_j^t := \tau_{j,\min}^t = \min_i \{p_{ij}^t\}, \qquad (8)$$

2) as the maximum value of the j coordinates of the vectors $\mathbf{p}_i^t$

$$\tau_j^t := \tau_{j,\max}^t = \max_i \{p_{ij}^t\}. \qquad (9)$$

3) as the mean value of the j coordinates of the vectors $\mathbf{p}_i^t$

$$\tau_j^t := \overline{\tau}_j^t = \frac{1}{m} \sum_{i=1}^{n} p_{ij}^t, \qquad (10)$$

4) all coordinates of the vector $\mathcal{T}^t$ are the same (given by the same function of t)

$$\tau_{j_1}^t = \tau_{j_2}^t > 0, \quad j_1, j_2 = \overline{1, n}. \qquad (11)$$

The limiting states

Theorem 2.

Let the coordinates of the vector $\mathcal{T}^t$ given by one of the equalities (8), (9), (10), (11). *Then every trajectory of the dynamical system* (3) *with initial state* $\{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_m\}$ *converges to a fixed point* $\{\mathbf{p}_1^\infty, \mathbf{p}_2^\infty, \ldots, \mathbf{p}_m^\infty\}$

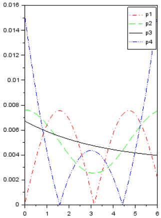
$$\mathbf{p}_i^\infty = \lim_{t\to\infty} \mathbf{p}_i^t, \ \forall\ i = \overline{1, m}.$$

Moreover, all limit vectors $\mathbf{p}_i^\infty$ *coincide with the vector* $\mathbf{w}$, *that is*

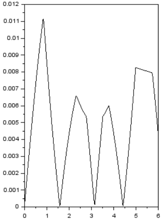
$$p_{ij}^\infty = \frac{\mathcal{T}_j}{W} \quad \forall\ i = \overline{1, m}, \quad j = \overline{1, n}. \tag{12}$$

Proposition 1.

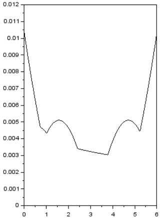
Limit state $\{\mathbf{p}_1^\infty, \mathbf{p}_2^\infty, \ldots, \mathbf{p}_m^\infty\}$ *is stable only if the coordinates of the vector* $\mathcal{T}^t$ *given by the equality* (11). *Then all limit coordinates are equal to* $\frac{1}{n}$. *In all other cases, the limiting state is unstable.*



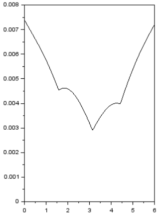
1) $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4\}$



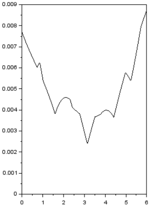
2) $\tau_j^t := \tau_{j,\min}^t$



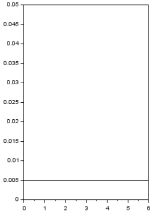
3) $\tau_j^t := \tau_{j,\max}^t$



4) $\tau_j^t = \overline{\tau}_j^t$

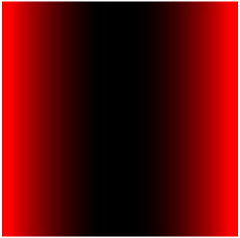


5) $\tau_j^t := \tau_{j,\min}^t + \tau_{j,\max}^t$

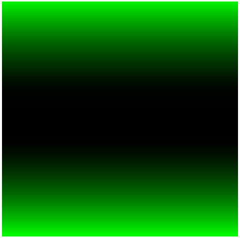


6) $\tau^t := \sum_{j=1}^{3} \tau_{j,\max}^t$

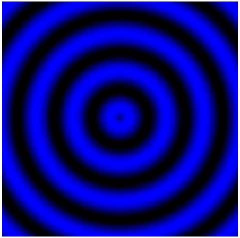
Fig.1. $m = 4, n = 200$, 1) $f_1(x) = 3|\sin(x)|$, $f_2(x) = \cos(x) + 2$, $f_3(x) = \left(\frac{3}{4}\right)^x + 1$, $f_4(x) = -(x-3)^2 + 2$



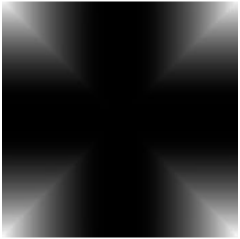
1) $\mathbf{p}_1$



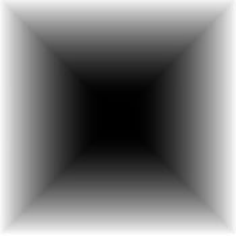
2) $\mathbf{p}_2$



3) $\mathbf{p}_3$



4) $\tau_j^t = \min_i \{p_{ij}^t\}$



5) $\tau_j^t = \max_i \{p_{ij}^t\}$

6) $\tau_j^t = \frac{1}{m}\sum_{i=1}^{m} p_{ij}^t$

Fig.2. 1)–3) Initial vector projections $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, $f_1(x,y) = y^2$, $f_2(x,y) = x^2$, $f_3(x,y) = 1 + \sin\sqrt{x^2+y^2}$, де $-20 \le x \le 20$, $-20 \le y \le 20$.

We define $\tau_j^t = \tau_j(t)$, $j = \overline{1,n}$ as a set of non-negative periodic functions with commensurable periods. Then, due to the property of periodic functions $W^t = W(t)$, $w_j^t = w_j(t) = \frac{\tau_j(t)}{W(t)}$ are also periodic with a principal period T.

Theorem 3.

Let all vectors $\mathbf{w}^t$ be invariant with time t, that is $\mathbf{w}^t = \mathbf{w}$, and their coordinates are equal to some non-negative constants c_j, $j = 1, \ldots, m$. Then every trajectory of the dynamical system (3) *with initial state $\{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_m\}$ converges to a fixed point $\{\mathbf{p}_1^\infty, \mathbf{p}_2^\infty, \ldots, \mathbf{p}_m^\infty\}$*

$$\mathbf{p}_i^\infty = \lim_{t\to\infty} \mathbf{p}_i^t = \mathbf{w}, \quad \forall\, i = 1, \ldots, m.$$

Theorem 4.

Let the period of function $w_j(t)$ *in the equations* (4) *is a positive integer* $T > 1$. *Then every trajectory of the dynamical system* (3) *with initial state* $\{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_m\}$ *converges to* ω*-limit set* $\mathbf{\Gamma}^\infty$*, which is a cyclic orbit. Moreover, the set* $\mathbf{\Gamma}^\infty$ *is invariant under the mapping* $\divideontimes$ *and consists of* T *ordered vectors* Γ_l, $l = 1, \ldots, T$

$$\Gamma_1 \xrightarrow{\divideontimes} \Gamma_2 \xrightarrow{\divideontimes} \Gamma_3 \xrightarrow{\divideontimes} \cdots \xrightarrow{\divideontimes} \Gamma_T \xrightarrow{\divideontimes} \Gamma_1.$$

ω*-limit set* $\mathbf{\Gamma}^\infty$ *is unstable.*

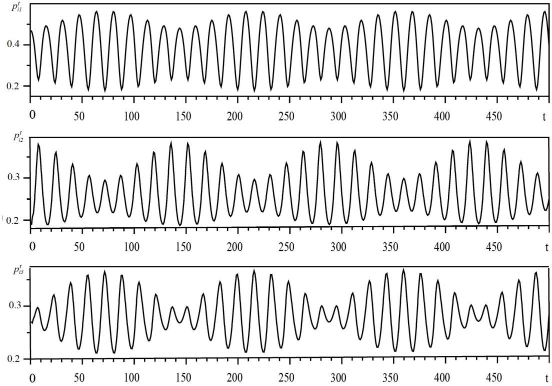


Fig.3. $\tau_1(t) = 5\cos(\frac{\pi t}{8}) + 8,\ \tau_2(t) = 3\sin(\pi t) + 5,\ \tau_3(t) = 2\cos(\frac{\pi t}{9}) + 7,$ T=144.

References

1. Koshmanenko V. D. The Spectral Theory of Conflict Dynamical Systems. — K.: Naukova dumka, 2016. – 287 p.
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Thank you!