On linear inhomogeneous boundary-value problems for differential systems in Sobolev spaces

Olena Atlasiuk

Kyiv, Institute of Mathematics,
National Academy of Sciences of Ukraine

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Introduction

We consider

the characteristics of solvability and continuity in a parameter of solutions of the most general (generic) classes of one-dimensional inhomogeneous boundary-value problems for systems of linear ordinary differential equations of an arbitrary order in Sobolev spaces on a finite interval.

The topic is actively engaged in such mathematicians as:

Boichuk,
Kiguradze, Ashordia
Mikhailets, Murach

Fredholm one-dimensional boundary-value problems

Let a finite interval $(a,b) \subset \mathbb{R}$ and parameters $\{m, n, r, l\} \subset \mathbb{N}$, $1 \leq p \leq \infty$, be given.

Linear boundary-value problem

$$(Ly)(t) := y^{(r)}(t) + \sum_{j=1}^{r} A_{r-j}(t) y^{(r-j)}(t) = f(t), \quad t \in (a,b), \tag{1}$$

$$By = c. \tag{2}$$

Here matrix-valued functions $A_{r-j}(\cdot) \in (W_p^n)^{m \times m}$, vector-valued function $f(\cdot) \in (W_p^n)^m$, vector $c \in \mathbb{C}^l$, linear continuous operator

$$B\colon (W_p^{n+r})^m \to \mathbb{C}^l \tag{3}$$

are arbitrarily chosen; vector-valued function $y(\cdot) \in (W_p^{n+r})^m$ is unknown.

The solutions of equation (1) fill the space $(W_p^{n+r})^m$ if its right-hand side $f(\cdot)$ runs through the space $(W_p^n)^m$. Hence, the condition (2) with operator (3) is generic condition for this equation.

It includes all known types of classical boundary conditions and numerous nonclassical conditions containing the derivatives (in general fractional) $y^{(k)}(\cdot)$ with $0 < k \leq n + r$.

Sobolev spaces

Complex Sobolev space $W_p^{n+r} := W_p^{n+r}\big([a,b];\mathbb{C}\big)$

$$W_p^{n+r}\big([a,b];\mathbb{C}\big) := \big\{y \in C^{n+r-1}[a,b] : y^{(n+r-1)} \in AC[a,b],\, y^{(n+r)} \in L_p[a,b]\big\}$$

This space is Banach relative to the norm

$$\|y\|_{n+r,p} = \sum_{k=0}^{n+r-1} \big\|y^{(k)}\big\|_p + \big\|y^{(n+r)}\big\|_p,$$

where $\|\cdot\|_p$ is the norm in $L_p\big([a,b];\mathbb{C}\big)$.
By $\|\cdot\|_{n+r,p}$, we also denote the norms in Banach spaces

$$(W_p^{n+r})^m := W_p^{n+r}([a,b];\mathbb{C}^m) \quad \text{and} \quad (W_p^{n+r})^{m\times m} := W_p^{n+r}([a,b];\mathbb{C}^{m\times m}).$$

They consist of the vector-valued functions and matrix-valued functions, respectively, all components of which belong to W_p^{n+r}.

Fredholm boundary-value problem and its index

With problem (1), (2), we associate the linear operator

$$(L,B)\colon (W_p^{n+r})^m \to (W_p^n)^m \times \mathbb{C}^l. \qquad (4)$$

A linear continuous operator $T\colon X \to Y$, where X and Y are Banach spaces, is called a Fredholm operator if its kernel $\ker T$ and cokernel $Y/T(X)$ are finite-dimensional. If this operator is Fredholm, then its range $T(X)$ is closed in Y and the index is finite:

$$\operatorname{ind} T := \dim\ker T - \dim(Y/T(X)) \in \mathbb{Z}.$$

Theorem 1.

The linear operator (4) is a bounded Fredholm operator with index $mr - l$.

Family of matrix Cauchy problems with the initial conditions

$$Y_k^{(r)}(t) + \sum_{j=1}^{r} A_{r-j}(t) Y_k^{(r-j)}(t) = O_m, \quad t \in (a,b),$$

$$Y_k^{(j-1)}(a) = \delta_{k,j} I_m, \qquad j \in \{1,\dots,r\}.$$

Solvability of the boundary-value problem

By $[BY_k]$, we denote the numerical $m\times l$ matrix, in which j-th column is result of the action of B on j-th column of $Y_k(\cdot)$.

Definition 1.

A block numerical matrix

$$\mathrm{M(L,B)} := ([BY_0],\dots,[BY_{r-1}]) \in \mathbb{C}^{mr\times l} \tag{5}$$

is characteristic matrix to problem (1), (2). It consists of r rectangular block columns $[BY_k(\cdot)]\in\mathbb{C}^{m\times l}$.

Theorem 2.

The dimensions of kernel and cokernel of the operator (4) *are equal to the dimensions of kernel and cokernel of matrix* (5)*, respectively:*

$$\dim\ker(L,B) = \dim\ker\big(M(L,B)\big),$$
$$\dim\operatorname{coker}(L,B) = \dim\operatorname{coker}\big(M(L,B)\big).$$

Corollary 1.

The operator (4) *is invertible if and only if* $l=mr$ *and the matrix* $M(L,B)$ *is nondegenerate.*

Consider problem (1), (2) putting $A(t) \equiv 0$ with the next boundary conditions:

$$By = \sum_{k=0}^{n-1} \alpha_k y^{(k)}(a) + \int_a^b \Phi(t) y^{(n)}(t)\mathrm{d}t, \quad y(\cdot) \in (W_p^n)^m.$$

Then we have

$$BY = \sum_{s=0}^{n-1} \alpha_s Y^{(s)}(a) + \int_a^b \Phi(t) Y^{(n)}(t)\mathrm{d}t, \quad Y(\cdot) = I_m,$$

$$M(L,B) = \alpha_0.$$

The numerical matrix α_0 does not depend on p, $\alpha_1, \ldots, \alpha_{n-1}$, and $\Phi(\cdot)$. Thus, the statement of Theorem 2 holds:

$$\dim\ker(M(L,B)) = \dim\ker(\alpha_0),$$
$$\dim\operatorname{coker}(M(L,B)) = \dim\operatorname{coker}(\alpha_0).$$

Application

Boundary-value problems depending on the parameter $k \in \mathbb{N}$

$$L(k)y(t,k) := y^{(r)}(t,k) + \sum_{j=1}^{r} A_{r-j}(t,k)y^{(r-j)}(t,k) = f(t,k), \quad t \in (a,b), \tag{6}$$

$$B(k)y(\cdot,k) = c(k), \quad k \in \mathbb{N}, \tag{7}$$

where $A_{r-j}(\cdot,k)$, $f(\cdot,k)$, $c(k)$, and linear continuous operator $B(k)$ satisfy the above conditions to problem (1), (2).

The sequence of linear continuous operators

$$(L(k),B(k)) \colon (W_p^{n+r})^m \to (W_p^n)^m \times \mathbb{C}^l,$$

and characteristic matrices

$$M\big(L(k),B(k)\big) := \big([B(k)Y_0(\cdot,k)],\ldots,[B(k)Y_{r-1}(\cdot,k)]\big) \subset \mathbb{C}^{mr\times l}.$$

Theorem 3.

If the sequence of operators $\big(L(k),B(k)\big)$ converges strongly to the operator $\big(L,B\big)$ then the sequence of characteristic matrices $M\big(L(k),B(k)\big)$ converges to the matrix $M\big(L,B\big)$ for $k \to \infty$.

Application

Corollary 2.

Under assumptions in Theorem 3, the following inequalities hold starting with sufficiently large k:

$$\dim\ker\left(L(k), B(k)\right) \leq \dim\ker\left(L, B\right),$$

$$\dim\operatorname{coker}\left(L(k), B(k)\right) \leq \dim\operatorname{coker}\left(L, B\right).$$

In particular, for sufficiently large k, we have:

1) if $l = mr$ and operator (L, B) is invertible, then the operators $(L(k), B(k))$ are also invertible;
2) if problem (1), (2) has a solution, then problems (6), (7) also have a solution;
3) if problem (1), (2) has a unique solution, then problems (6), (7) also have a unique solution [1, 3, 4].

Parameterized boundary-value problem

Boundary-value problem depending on a parameter $\varepsilon \in [0, \varepsilon_0)$

$$L(\varepsilon)y(t,\varepsilon) := y^{(r)}(t,\varepsilon) + \sum_{j=1}^{r} A_{r-j}(t,\varepsilon) y^{(r-j)}(t,\varepsilon) = f(t,\varepsilon), \ t \in (a,b), \quad (8)$$

$$B(\varepsilon)y(\cdot;\varepsilon) = c(\varepsilon), \quad (9)$$

where a linear continuous operator

$$B(\varepsilon)\colon (W_p^{n+r})^m \to \mathbb{C}^{rm}.$$

According to Theorem 1, problem (8), (9) is a Fredholm one with zero index for every $\varepsilon \in [0, \varepsilon_0)$.

Definition 2.

The solution to the problem (8), (9) depends continuously on a parameter ε at $\varepsilon = 0$ if the conditions are satisfied:

($*$) there exists a positive number $\varepsilon_1 < \varepsilon_0$ such that, for any $\varepsilon \in [0, \varepsilon_1)$ and arbitrary chosen $f(\cdot;\varepsilon) \in (W_p^n)^m$, $c(\varepsilon) \in \mathbb{C}^{rm}$, this problem has a unique solution $y(\cdot;\varepsilon) \in (W_p^{n+r})^m$;

($**$) the convergence of right-hand sides $f(\cdot;\varepsilon) \to f(\cdot;0)$ and $c(\varepsilon) \to c(0)$ implies the convergence of solutions

$$y(\cdot;\varepsilon) \to y(\cdot;0) \quad \text{in} \quad (W_p^{n+r})^m \quad \text{as} \quad \varepsilon \to 0+.$$

Criterion of continuous dependence on a parameter

Consider the following conditions:

(0) the homogeneous boundary-value problem

$$L(0)y(t,0)=0, \quad t\in(a,b), \quad B(0)y(\cdot,0)=0$$

has only the trivial solution;

(I) $A_{r-j}(\cdot;\varepsilon)\to A_{r-j}(\cdot;0)$ in $(W_p^n)^{m\times m}$ for every $j\in\{1,\dots,r\}$;

(II) $B(\varepsilon)y\to B(0)y$ in $\mathbb{C}^{rm}$ for every $y\in(W_p^{n+r})^m$.

Theorem 4.

The solution to the problem (8), (9) depends continuously on the parameter ε at $\varepsilon=0$ if and only if this problem satisfies Conditions (0), (I), and (II).

Degree of convergence of the solutions

We supplement our result with a two-sided estimate of the error $\left\|y(\cdot;0)-y(\cdot;\varepsilon)\right\|_{n+r,p}$ of solution $y(\cdot;\varepsilon)$ via its discrepancy

$$\widetilde{d}_{n,p}(\varepsilon) := \left\|L(\varepsilon)y(\cdot;0)-f(\cdot;\varepsilon)\right\|_{n,p} + \left\|B(\varepsilon)y(\cdot;0)-c(\varepsilon)\right\|_{\mathbb{C}^{rm}}.$$

Here, we interpret $y(\cdot;0)$ as an approximate solution to problem (8), (9).

Theorem 5.

Let the problem (8), (9) satisfies Conditions (0), (I), and (II). Then there exist positive numbers $\varepsilon_2 < \varepsilon_1$, γ_1, and γ_2, such that

$$\gamma_1\,\widetilde{d}_{n,p}(\varepsilon) \le \left\|y(\cdot;0)-y(\cdot;\varepsilon)\right\|_{n+r,p} \le \gamma_2\,\widetilde{d}_{n,p}(\varepsilon)$$

for any $\varepsilon \in (0,\varepsilon_2)$. Here, the numbers ε_2, γ_1, and γ_2 do not depend on $y(\cdot;0)$, and $y(\cdot;\varepsilon)$.

Thus, the error and discrepancy of the solution to problem (8), (9) are of the same degree of smallness [2, 6, 7].

Multi-point boundary-value problem

For any $\varepsilon \in [0, \varepsilon_0)$, $\varepsilon_0 > 0$, we associate with the system (8)

multi-point Fredholm boundary condition

$$B(\varepsilon)y(\cdot, \varepsilon) = \sum_{j=0}^{N} \sum_{k=1}^{\omega_j(\varepsilon)} \sum_{l=0}^{n+r-1} \beta_{j,k}^{(l)}(\varepsilon) y^{(l)}(t_{j,k}(\varepsilon), \varepsilon) = q(\varepsilon), \qquad (10)$$

where the numbers $\{N, \omega_j(\varepsilon)\} \subset \mathbb{N}$, vectors $q(\varepsilon) \in \mathbb{C}^{rm}$, matrices $\beta_{j,k}^{(l)}(\varepsilon) \in \mathbb{C}^{m \times m}$, and points $\{t_j, t_{j,k}(\varepsilon)\} \subset [a, b]$ are arbitrarily given.

Sufficient constructive conditions are established under which the solutions to the problem (8), (10) are continuous with respect to the parameter ε at $\varepsilon = 0$ in W_p^{n+r}, $1 \leq p \leq \infty$ [5, 8].

Publication of results

1. *Atlasiuk, O. M.; Mikhailets, V. A.* Fredholm one-dimensional boundary-value problems in Sobolev spaces. Ukrainian Math. J. 70 (2019), no. 10, 1526–1537. *DOI: 10.1007/s11253-019-01588-w*
2. *Atlasiuk, O. M.; Mikhailets, V. A.* Fredholm one-dimensional boundary-value problems with parameter in Sobolev spaces. Ukrainian Math. J. 70 (2019), no. 11, 1677–1687. *DOI: 10.1007/s11253-019-01599-7*
3. *Atlasiuk, O. M.; Mikhailets, V. A.* On solvability of inhomogeneous boundary-value problems in Sobolev spaces. Dopov. Nac. akad. nauk Ukr. (2019), no. 11, 3–7. *DOI: 10.15407/dopovidi2019.11.003*
4. *Atlasiuk, O.; Mikhailets, V.* On linear boundary-value problems for differential systems in Sobolev spaces. International Workshop on the Qualitative Theory of Differential Equations "QUALITDE – 2019", Tbilisi, Georgia: Abstracts, 19–22.

Publication of results

5. *Atlasiuk, O. M.* Limit theorems for solutions of multipoint boundary-value problems in Sobolev spaces. J. Math. Sci. 247 (2020), no. 2, 238–247. *DOI: 10.1007/s10958-020-04799-w*
6. *Atlasiuk, O. M.; Mikhailets, V. A.* On Fredholm parameter-dependent boundary-value problems in Sobolev spaces. Dopov. Nac. akad. nauk Ukr. (2020), no. 6, 3–6. *DOI: 10.15407/dopovidi2020.06.003*
7. *Atlasiuk, O.; Mikhailets, V.* On generic inhomogeneous boundary-value problems for differential systems in Sobolev spaces. International Workshop on the Qualitative Theory of Differential Equations ”QUALITDE – 2020”, Tbilisi, Georgia: Abstracts, 22–26.
8. *Atlasiuk, O. M.* Limit theorems for solutions of multipoint boundary-value problems with a parameter in Sobolev spaces. Ukrainian Math. J. 72 (2021), no. 8, 1175–1184. *DOI: 10.1007/s11253-020-01859-x*