Eigenfunctions of the Star graph for all non-zero eigenvalues

Vladislav Kabanov

Krasovskii Institute of Mathematics and Mechanics
Yekaterinburg, Russia

Base on joint work with E. Konstantinova, L. Shalaginov, and A. Valyuzhenich

8ECM June 24, 2021
Graphs and Groups, Geometries and GAP

Let $\Gamma = (V, E)$ be a simple graph. We denote by $\Gamma(x)$ the neighbourhood of a vertex x. Let θ be an eigenvalue of the adjacency matrix of Γ.

Definition

A non-zero function $f : V \longrightarrow \mathbb{R}$ is called a **θ-eigenfunction** of Γ if for any vertex x of Γ the following condition is true

$$\theta \cdot f(x) = \sum_{y \in \Gamma(x)} f(y).$$

Let G be a finite group and S be a generating set of G. Let S not contain the identity and be closed with respect to inversion.

Definition

The **Cayley graph** $\Gamma = \mathbb{C}\text{ay}(G, S)$ of G generated by S is a graph for which:

- the vertex set V is identified with G,
- the edge set E is $\{\ (x, sx) \mid \text{for any } x \in G \text{ and for any } s \in S\ \}$.

For any vertex x of Γ the neighbourhood of x in Γ is

$$\Gamma(x) = Sx = \{\ sx \mid s \in S\ \}.$$

The Star graph $\mathbb{S}_n$

Let Ω be the set $\{ 1, \ldots, n \}$, $n \geq 2$.

We consider the symmetric group Sym_Ω and put

$$S = \{ (1\ 2), (1\ 3), \ldots, (1\ n) \}.$$

Definition

The **Star graph** $\mathbb{S}_n$ is the Cayley graph over the symmetric group Sym_Ω with the generating set S.

The Star graph is interesting as network topology, because it is an attractive alternative to the hypercube, a popular network for interconnecting processors in a parallel computer, and it compares favorably with hypercube in several aspects.

R. Ko'tter and F. R. Kschischang. Coding for errors and erasures in random network coding. IEEE Trans. Inf. Theory, 54, 8 (2008) 3579–3591.

In this talk, we are primarily interested in the eigenfunctions and the spectrum of the Star graph $\mathbb{S}_n$.

It was shown by Krakovski and Mohar that the spectrum of the Star graph $\mathbb{S}_n$ contains all integers from $-n+1$ to $n-1$ (except 0 if $n=2$ or $n=3$).

Since the Star graph is bipartite, $\text{mul}(n-i) = \text{mul}(-n+i)$ for any integer i, where $1 \leq i \leq n$. Moreover, $\pm(n-1)$ are simple eigenvalues of $\mathbb{S}_n$.

R. Krakovski, B. Mohar, Spectrum of Cayley graphs on the symmetric group generated by transposition, Linear Algebra and its Applications, ***437*** *(2012) 1033–1039.*

Group algebra $\mathbb{C}[G]$

Let $\mathbb{C}$ be a complex field and let $\mathbb{C}[G]$ be the group algebra of G over $\mathbb{C}$.

For a subset S in G, consider the element $\underline{S} \in \mathbb{C}[G]$ given by

$$\underline{S} = \sum_{s \in S} s.$$

Left multiplication of elements from $\mathbb{C}[G]$ by $\underline{S}$ is a linear transformation of $\mathbb{C}[G]$.

It is known that the matrix of this linear transformation coincides with the adjacency matrix of $\mathbb{C}\mathrm{ay}(G, S)$.

Hence, spectral theory of Cayley graphs is connected with the theory of group representations.

The spectrum of the Star graph $\mathbb{S}_n$

In [CF2012], Chapuy and Feray pointed out that the spectrum of the Star graph $\mathbb{S}_n$ can be found using the Jucys theory. In [J1974], Jucys regarded the action of the element equal sum of transpositions

$$J_n = (1\ n) + (2\ n) + \ldots + (n-1\ n)$$

on the complex group algebra over symmetric group $\mathbb{C}[Sym_n]$ by left multiplication.

We have to swap 1 and n, in these transpositions to obtain the generating set of the Star graph S_n.

[CF2012] *G. Chapuy, V. Feray, A note on a Cayley graph of* Sym_n*, arXiv:1202.4976v2 (2012) 1–3.*
[J1974] *A. Jucys, Symmetric polynomials and the center of the symmetric group ring, Reports Math. Phys.,* **5** *(1974) 107–112.*

However, eigenfunctions corresponding to the eigenvalues of the Star graph $\mathbb{S}_n$ were never obtained explicitly.

In this talk we remind the definition of PI-eigenfunctions of the Star graph $\mathbb{S}_n$, $n \geq 3$ for any eigenvalue $n - m - 1$, where $n > 2m > 0$ from [1],

and we present a generalization of this family for all non-zero eigenvalues from [2].

[1] *S. Goryainov, V. Kabanov, E. Konstantinova, L. Shalaginov, A. Valyuzhenich,* $\mathbb{PI}$*-eigenfunctions of the Star graphs, Linear Algebra and its Applications, 586 (2020) 7–27.*

[2] *V. Kabanov, E. Konstantinova, L. Shalaginov, A. Valyuzhenich, The Star graph eigenfunctions with non-zero eigenvalues, Linear Algebra and its Applications 610, (2021) 222–226.*

Let's define two m-tuples:

- $P_m = ((j_1, k_1), (j_2, k_2), \ldots, (j_m, k_m))$, where $2m$ pairwise distinct elements from the set $\{ 2, \ldots, n \}$ are arranged into m pairs.
- $I_m = (i_1, i_2, \ldots, i_m)$ of m pairwise distinct elements from the set $\{ 1, \ldots, n \}$;

Let $\pi = [\pi_1 \pi_2 \ldots \pi_n]$ be a permutation from Sym_n.

For any t in $\{ 2, \ldots, n \}$ one of the following statements hold:

$$\pi_{j_t} = i_t,$$

$$\pi_{k_t} = i_t,$$

$$\pi_{j_t} \neq i_t \text{ and } \pi_{k_t} \neq i_t.$$

Let's define two m-tuples:

- $P_m = ((j_1, k_1), (j_2, k_2), \ldots, (j_m, k_m))$, where $2m$ pairwise distinct elements from the set $\{ 2, \ldots, n \}$ are arranged into m pairs.
- $I_m = (i_1, i_2, \ldots, i_m)$ of m pairwise distinct elements from the set $\{ 1, \ldots, n \}$;

Let $\pi = [\pi_1 \pi_2 \ldots \pi_n]$ be a permutation from Sym_n.

For any t in $\{ 2, \ldots, n \}$ one of the following statements hold:

$$\pi_{j_t} = i_t,$$

$$\pi_{k_t} = i_t,$$

$$\pi_{j_t} \neq i_t \text{ and } \pi_{k_t} \neq i_t.$$

If in π for any $t \in \{ 1, 2, \ldots, m \}$ either $\pi_{j_t} = i_t$ or $\pi_{k_t} = i_t$ hold, we define a vector $X_\pi = (x_1, x_2, \ldots, x_m)$, where

$$x_t = \begin{cases} 1, & \text{if } \pi_{j_t} = i_t, \\ 0, & \text{if } \pi_{k_t} = i_t. \end{cases}$$

Definition

We define the function $f = f_{I_m}^{P_m} : \mathrm{Sym}_n \to \mathbb{R}$ as follows.

- If there exists t such that $\pi_{j_t} \neq i_t$ and $\pi_{k_t} \neq i_t$, then $f = 0$.
- If for any t either $\pi_{j_t} = i_t$ or $\pi_{k_t} = i_t$, then

$$f = \begin{cases} 1, & \text{if } wt(X_\pi) \text{ is an even number}, \\ -1, & \text{if } wt(X_\pi) \text{ is an odd number}. \end{cases}$$

If in π for any $t \in \{ 1, 2, \dots, m \}$ either $\pi_{j_t} = i_t$ or $\pi_{k_t} = i_t$ hold, we define a vector $X_\pi = (x_1, x_2, \dots, x_m)$, where

$$x_t = \begin{cases} 1, & \text{if } \pi_{j_t} = i_t, \\ 0, & \text{if } \pi_{k_t} = i_t. \end{cases}$$

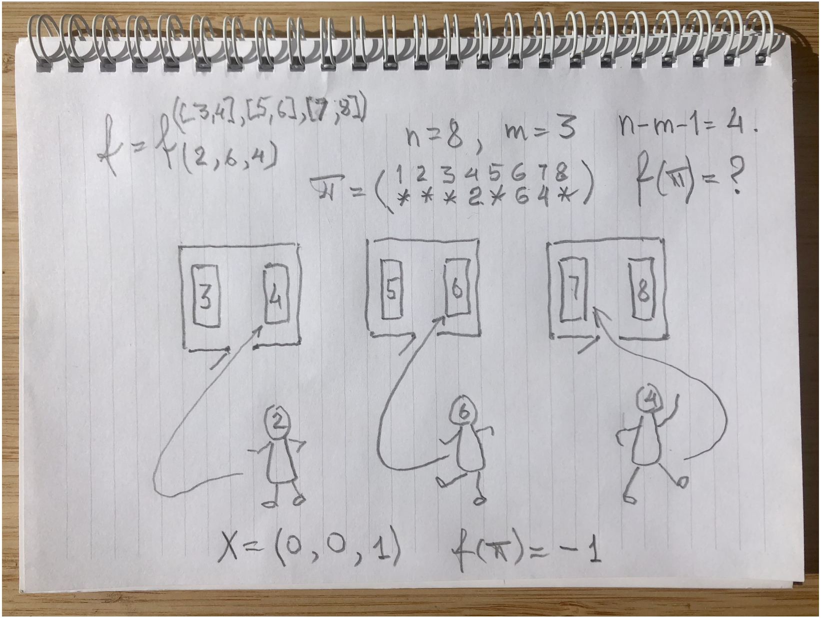
Definition

We define the function $f = f_{I_m}^{P_m} : \mathrm{Sym}_n \to \mathbb{R}$ as follows.

- If there exists t such that $\pi_{j_t} \neq i_t$ and $\pi_{k_t} \neq i_t$, then $f = 0$.
- If for any t either $\pi_{j_t} = i_t$ or $\pi_{k_t} = i_t$, then

$$f = \begin{cases} 1, & \text{if } wt(X_\pi) \text{ is an even number}, \\ -1, & \text{if } wt(X_\pi) \text{ is an odd number}. \end{cases}$$

$\mathbb{PI}$-eigenfunctions of the Star graph $\mathbb{S}_n$



$\mathbb{PI}$-eigenfunctions of the Star graph $\mathbb{S}_n$

Theorem

For $n \geq 3$ and $n > 2m > 0$ the function $f_{I_m}^{P_m}$ is an eigenfunction with eigenvalue $n - m - 1$ of the Star graph $\mathbb{S}_n$.

The eigenfunctions from this Theorem we call $\mathbb{PI}$-eigenfunctions.

Obviously, such a function cannot be defined for $m \geq n/2$.

Generalization of $\mathbb{PI}$-eigenfunctions

Further we present a generalization of this family of $\mathbb{PI}$-eigenfunctions of the Star graph $\mathbb{S}_n$, $n \geq 3$, for all its non-zero eigenvalues.

To present a generalization of $\mathbb{PI}$-eigenfunctions, we need some new definitions.

Definition of new eigenfunctions

Let Ω be the set $\{ 1, \ldots, n \}$, $n \geq 3$.

For any positive integers m and c such that $m < n - 1$ and $c < n - m - 1$, we define a function from Sym_Ω to $\mathbb{R}$ depending on the choice of two subsets Δ, Λ of Ω and two tuples P_Δ, I_Λ as follows.

Let Δ be an $(m + c)$-element subset of $\Omega \setminus \{ 1 \}$, and $\Delta_1, \ldots, \Delta_c$ be its partition with $|\Delta_t| = \delta_t$, where $1 \leq t \leq c$. For each t, let $P_t = (j_{t1}, \ldots, j_{t\delta_t})$ be a tuple consisting of all elements of the set Δ_t, i.e. all components of P_t are pairwise distinct. Denote $P_\Delta = (P_1, \ldots, P_c)$.

Let Λ be an m-element subset of Ω, and $\Lambda_1, \ldots, \Lambda_c$ be its partition with $|\Lambda_t| = \lambda_t = \delta_t - 1$, where $1 \leq t \leq c$. For each t, let $I_t = (i_{t1}, \ldots, i_{t\lambda_t})$ be a tuple consisting of all elements of the set Λ_t, i.e. all components of I_t are pairwise distinct. Denote $I_\Lambda = (I_1, \ldots, I_c)$.

Let Ω be the set $\{ 1, \ldots, n \}$, $n \geq 3$.

For any positive integers m and c such that $m < n - 1$ and $c < n - m - 1$, we define a function from Sym_Ω to $\mathbb{R}$ depending on the choice of two subsets Δ, Λ of Ω and two tuples P_Δ, I_Λ as follows.

Let Δ be an $(m+c)$-element subset of $\Omega \setminus \{ 1 \}$, and $\Delta_1, \ldots, \Delta_c$ be its partition with $|\Delta_t| = \delta_t$, where $1 \leq t \leq c$. For each t, let $P_t = (j_{t1}, \ldots, j_{t\delta_t})$ be a tuple consisting of all elements of the set Δ_t, i.e. all components of P_t are pairwise distinct. Denote $P_\Delta = (P_1, \ldots, P_c)$.

Let Λ be an m-element subset of Ω, and $\Lambda_1, \ldots, \Lambda_c$ be its partition with $|\Lambda_t| = \lambda_t = \delta_t - 1$, where $1 \leq t \leq c$. For each t, let $I_t = (i_{t1}, \ldots, i_{t\lambda_t})$ be a tuple consisting of all elements of the set Λ_t, i.e. all components of I_t are pairwise distinct. Denote $I_\Lambda = (I_1, \ldots, I_c)$.

Definition of new eigenfunctions

Let Ω be the set $\{ 1, \ldots, n \}$, $n \geq 3$.

For any positive integers m and c such that $m < n - 1$ and $c < n - m - 1$, we define a function from Sym_Ω to $\mathbb{R}$ depending on the choice of two subsets Δ, Λ of Ω and two tuples P_Δ, I_Λ as follows.

Let Δ be an $(m + c)$-element subset of $\Omega \setminus \{ 1 \}$, and $\Delta_1, \ldots, \Delta_c$ be its partition with $|\Delta_t| = \delta_t$, where $1 \leq t \leq c$. For each t, let $P_t = (j_{t1}, \ldots, j_{t\delta_t})$ be a tuple consisting of all elements of the set Δ_t, i.e. all components of P_t are pairwise distinct. Denote $P_\Delta = (P_1, \ldots, P_c)$.

Let Λ be an m-element subset of Ω, and $\Lambda_1, \ldots, \Lambda_c$ be its partition with $|\Lambda_t| = \lambda_t = \delta_t - 1$, where $1 \leq t \leq c$. For each t, let $I_t = (i_{t1}, \ldots, i_{t\lambda_t})$ be a tuple consisting of all elements of the set Λ_t, i.e. all components of I_t are pairwise distinct. Denote $I_\Lambda = (I_1, \ldots, I_c)$.

Let H be a subgroup of Sym_Ω such that $H =$

$= \{\, \tau \in \mathrm{Sym}_\Omega \,|\, \tau(\Delta_t) = \Delta_t, 1 \leq t \leq c, \text{ and } \tau(i) = i \text{ for any } i \in \Omega \backslash \Delta \,\},$

i.e. any permutation τ from H fixes all points from $\Omega \setminus \Delta$, and $\tau(\Delta_t) = \Delta_t$ means that $\tau(j) \in \Delta_t$ for any $j \in \Delta_t$.

If $\pi \in \mathrm{Sym}_\Omega$, then $\pi(P_t) = (\pi(j_{t1}), \ldots, \pi(j_{t\delta_t}))$.

For given $\pi \in \mathrm{Sym}_\Omega$ and $\tau \in H$, by the *precedence relation*

$$I_\Lambda \prec \pi(\tau(P_\Delta))$$

we mean that for any t, $1 \leq t \leq c$, and any $i_{ts} \in \Lambda_t$ there exists $j_{ts'} \in \Delta_t$ such that $i_{ts} = \pi(\tau(j_{ts'}))$, moreover $\tau \circ \pi$ preserves the ordering of any P_t. In other words, I_t can be obtained from $\pi(\tau(P_t))$ by deleting one component for any t.

Let H be a subgroup of Sym_Ω such that $H =$

$= \{ \tau \in \mathrm{Sym}_\Omega \,|\, \tau(\Delta_t) = \Delta_t, 1 \le t \le c, \text{ and } \tau(i) = i \text{ for any } i \in \Omega \backslash \Delta \},$

i.e. any permutation τ from H fixes all points from $\Omega \setminus \Delta$, and $\tau(\Delta_t) = \Delta_t$ means that $\tau(j) \in \Delta_t$ for any $j \in \Delta_t$.

If $\pi \in \mathrm{Sym}_\Omega$, then $\pi(P_t) = (\pi(j_{t1}), \ldots, \pi(j_{t\delta_t}))$.

For given $\pi \in \mathrm{Sym}_\Omega$ and $\tau \in H$, by the *precedence relation*

$$I_\Lambda \prec \pi(\tau(P_\Delta))$$

we mean that for any t, $1 \le t \le c$, and any $i_{ts} \in \Lambda_t$ there exists $j_{ts'} \in \Delta_t$ such that $i_{ts} = \pi(\tau(j_{ts'}))$, moreover $\tau \circ \pi$ preserves the ordering of any P_t. In other words, I_t can be obtained from $\pi(\tau(P_t))$ by deleting one component for any t.

For a given $\pi \in \mathrm{Sym}_\Omega$ and $\tau \in H$, the precedence relation does not hold for I_Λ and $\pi(\tau(P_\Delta))$ if and only if either there exists i_{ts} such that $i_{ts} \neq \pi(\tau(j_{ts'}))$ for some $j_{ts'} \in \Delta_t$ or $\tau \circ \pi$ does not preserve the ordering of some P_t. Let $f = f_{I_\Lambda}^{P_\Delta}$.

Definition

$$f(\pi) = \begin{cases} +1, & \text{if } I_\Lambda \prec \pi(\tau(P_\Delta)) \text{ for an even permutation } \tau \in H; \\ -1, & \text{if } I_\Lambda \prec \pi(\tau(P_\Delta)) \text{ for an odd permutation } \tau \in H; \\ 0, & \text{if } I_\Lambda \not\prec \pi(\tau(P_\Delta)) \text{ for any permutation } \tau \in H. \end{cases} \quad (1)$$

Definition of new eigenfunctions

For a given $\pi \in \mathrm{Sym}_\Omega$ and $\tau \in H$, the precedence relation does not hold for I_Λ and $\pi(\tau(P_\Delta))$ if and only if either there exists i_{ts} such that $i_{ts} \neq \pi(\tau(j_{ts'}))$ for some $j_{ts'} \in \Delta_t$ or $\tau \circ \pi$ does not preserve the ordering of some P_t. Let $f = f_{I_\Lambda}^{P_\Delta}$.

Definition

$$f(\pi) = \begin{cases} +1, & \text{if } I_\Lambda \prec \pi(\tau(P_\Delta)) \text{ for an even permutation } \tau \in H; \\ -1, & \text{if } I_\Lambda \prec \pi(\tau(P_\Delta)) \text{ for an odd permutation } \tau \in H; \\ 0, & \text{if } I_\Lambda \not\prec \pi(\tau(P_\Delta)) \text{ for any permutation } \tau \in H. \end{cases} \quad (1)$$

Example 1. Let $n = 7$ and $\Omega = \{ 1, 2, 3, 4, 5, 6, 7 \}$. We put $m = 3$, $c = 2$ and consider $\Delta = \{ 2, 4 \} \cup \{ 3, 5, 6 \}$, $\Lambda = \{ 1 \} \cup \{ 3, 4 \}$. For given tuples $P_\Delta = ((2, 4), (3, 5, 6))$ and $I_\Lambda = ((1), (3, 4))$, we find the values of $f_{I_\Lambda}^{P_\Delta}$ for some permutations π.

If $\pi = [7, 1, 4, 5, 3, 6, 2]$, then $\pi_2 = 1$, $\pi_4 = 5 \notin \Lambda_1$, $\pi_3 = 4$, $\pi_5 = 3$, $\pi_6 = 6 \notin \Lambda_2$. For odd permutation $\tau = (3\ 5)$ the precedence relation $\prec$ holds since $\pi(\tau(3)) = 3$ and $\pi(\tau(5)) = 4$, therefore, $f_{I_\Lambda}^{P_\Delta}(\pi) = -1$.

If $\pi = [1, 5, 4, 7, 3, 6, 2]$, then $\pi_2 = 5 \notin \Lambda_1$, $\pi_4 = 7 \notin \Lambda_1$, and hence $f_{I_\Lambda}^{P_\Delta}(\pi) = 0$.

Example 1. Let $n = 7$ and $\Omega = \{ 1, 2, 3, 4, 5, 6, 7 \}$. We put $m = 3$, $c = 2$ and consider $\Delta = \{ 2, 4 \} \cup \{ 3, 5, 6 \}$, $\Lambda = \{ 1 \} \cup \{ 3, 4 \}$. For given tuples $P_\Delta = ((2, 4), (3, 5, 6))$ and $I_\Lambda = ((1), (3, 4))$, we find the values of $f_{I_\Lambda}^{P_\Delta}$ for some permutations π.

If $\pi = [7, 1, 4, 5, 3, 6, 2]$, then $\pi_2 = 1$, $\pi_4 = 5 \notin \Lambda_1$, $\pi_3 = 4$, $\pi_5 = 3$, $\pi_6 = 6 \notin \Lambda_2$. For odd permutation $\tau = (3\ 5)$ the precedence relation $\prec$ holds since $\pi(\tau(3)) = 3$ and $\pi(\tau(5)) = 4$, therefore, $f_{I_\Lambda}^{P_\Delta}(\pi) = -1$.

If $\pi = [1, 5, 4, 7, 3, 6, 2]$, then $\pi_2 = 5 \notin \Lambda_1$, $\pi_4 = 7 \notin \Lambda_1$, and hence $f_{I_\Lambda}^{P_\Delta}(\pi) = 0$.

Example 1. Let $n = 7$ and $\Omega = \{ 1, 2, 3, 4, 5, 6, 7 \}$. We put $m = 3$, $c = 2$ and consider $\Delta = \{ 2, 4 \} \cup \{ 3, 5, 6 \}$, $\Lambda = \{ 1 \} \cup \{ 3, 4 \}$. For given tuples $P_\Delta = ((2, 4), (3, 5, 6))$ and $I_\Lambda = ((1), (3, 4))$, we find the values of $f_{I_\Lambda}^{P_\Delta}$ for some permutations π.

If $\pi = [7, 1, 4, 5, 3, 6, 2]$, then $\pi_2 = 1$, $\pi_4 = 5 \notin \Lambda_1$, $\pi_3 = 4$, $\pi_5 = 3$, $\pi_6 = 6 \notin \Lambda_2$. For odd permutation $\tau = (3\ 5)$ the precedence relation $\prec$ holds since $\pi(\tau(3)) = 3$ and $\pi(\tau(5)) = 4$, therefore, $f_{I_\Lambda}^{P_\Delta}(\pi) = -1$.

If $\pi = [1, 5, 4, 7, 3, 6, 2]$, then $\pi_2 = 5 \notin \Lambda_1$, $\pi_4 = 7 \notin \Lambda_1$, and hence $f_{I_\Lambda}^{P_\Delta}(\pi) = 0$.

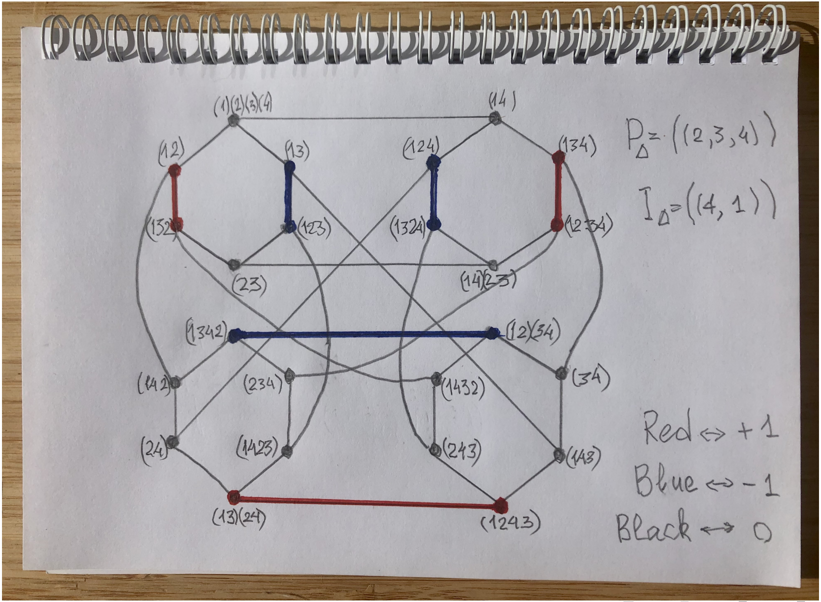
Main Theorem

Theorem

For any integers n and m, where $n \geq 3$ and $m < n - 1$, the function $f_{I_\Lambda}^{P_\Delta}$ is an $(n - m - 1)$-eigenfunction of the Star graph $\mathbb{S}_n$.

Example 2. Let $n = 4$, $m = 2$, $c = 1$. For the tuples $P_\Delta = ((2,3,4))$ and $I_\Lambda = ((4,1))$, the function $f_{I_\Lambda}^{P_\Delta}$ is an 1-eigenfunction of the Star graph $\mathbb{S}_4$.

Note that there are no PI-eigenfunctions for the eigenvalue 1 of the Star graph $\mathbb{S}_4$.


(1)(2)(3)(4)
(14)
(12)
(13)
(124)
(134)
(132)
(123)
(1324)
(1234)
(23)
(14)(23)
(1342)
(12)(34)
(142)
(234)
(1432)
(34)
(24)
(1423)
(243)
(143)
(13)(24)
(1243)
P_Δ = ((2,3,4))
I_Δ = ((4, 1))
Red ⇔ +1
Blue ⇔ −1
Black ⇔ 0

Remark 1.
Since the Star graph is bipartite, the $(-n+m+1)$-eigenfunction is known whenever the $(n-m-1)$-eigenfunction is known.

Remark 2.
If $|\Lambda_t| = 1$, where $1 \leq t \leq c$, then the new eigenfunction is a PI-eigenfunction.

Thank you for your attention!