Partitioning the projective plane and the dunce hat

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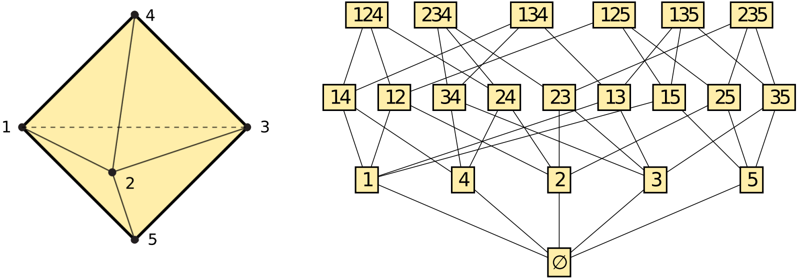
Portorož - Slovenia

Partitionability



A complex and its face poset. Source: (Duval, Klivans and Martin, 2017)

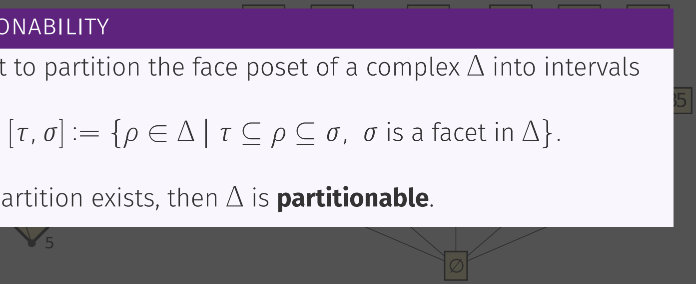
Partitionability

Partitionability

We want to partition the face poset of a complex Δ into intervals

$$[\tau, \sigma] := \{\rho \in \Delta \mid \tau \subseteq \rho \subseteq \sigma,\ \sigma \text{ is a facet in } \Delta\}.$$

If that partition exists, then Δ is **partitionable**.



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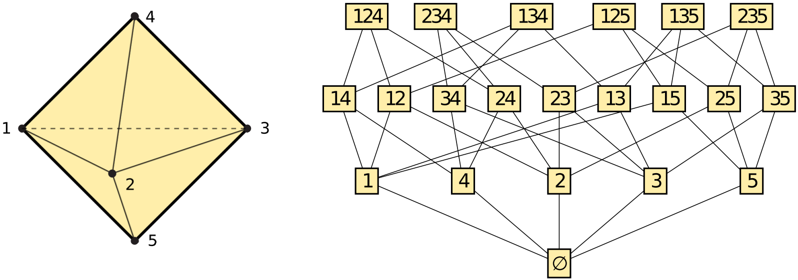
Partitionability



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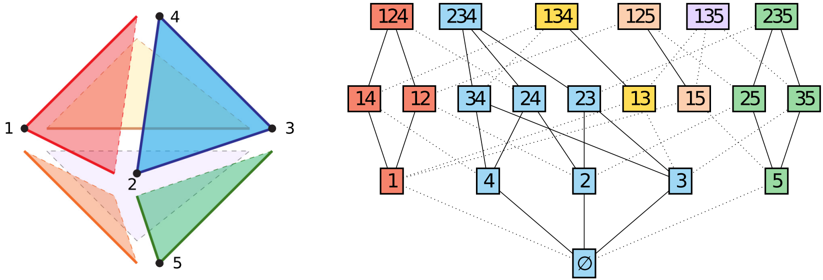

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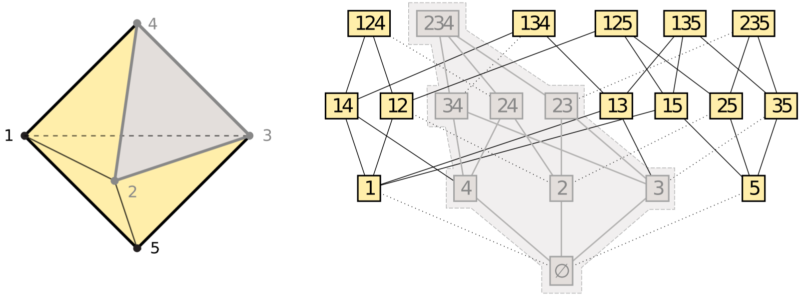
Partitionability



A **relative** complex and its face poset. Source: (Duval, Klivans and Martin, 2017)

Why does partitionability matter?

1 Face counting problems

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2. Computational issues

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2. Computational issues
3. Interplay between properties

It was believed that

Cohen–Macaulay $\Longrightarrow$ partitionable. (Stanley, 1979), (Garsia, 1980)

However

Cohen–Macaulay $\not\Longrightarrow$ partitionable. (Duval et al., 2016)

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What can we say about 2-dim complexes?

In our work...

2 We show that the following 2-dim spaces are partitionable:

- The real projective plane.
- The dunce hat.

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- The real projective plane.
- The dunce hat.

In our work...

1. We devised simple tools to get partitioning schemes of a (relative) complex in terms of constituent (relative) subcomplexes.

2. We show that the following 2-dim spaces are partitionable:
   - The open Möbius strip.
   - The real projective plane.
   - The dunce hat.

Toolbox

STRATEGY — DIVIDE-AND-CONQUER-*ISH*

The "tearing-apart-then-gluing-back" method

- Divide the complex into partitionable subcomplexes.
- Glue back.

Toolbox

Strategy — Divide-and-conquer-*ish*

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Our gluing tools

Name	Input (# r.s.c.)*	Preserves shellability
Shelling-Like Lemma	2	True
Folding operation	1	False

* Relative simplicial complexes

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Strategy — Divide-and-conquer-*ish*

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* Relative simplicial complexes

Facts on disks (a.k.a. 2-balls)

- (Relative) shellability $\Longrightarrow$ (Relative) partitionability.
- (Relative) disks (+conditions in the boundary) are shellable.

Möbius strip

Theorem (SG, 2020+)

Any triangulation of the Möbius strip M is partitionable relative to its own boundary.

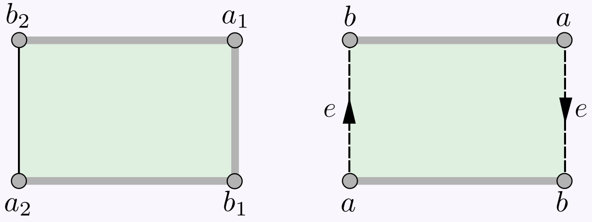
Möbius strip

Theorem (SG, 2020+)

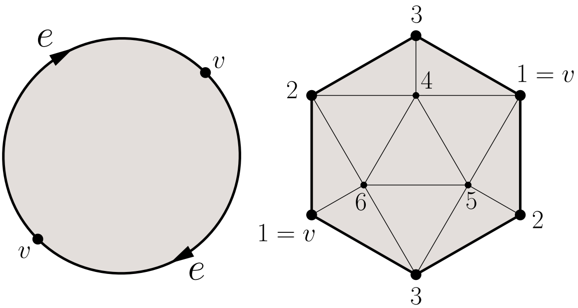
Any triangulation of the Möbius strip M is partitionable relative to its own boundary.

Proof outline:

- Start with a simplicial triangulation Δ_M of M.
- Cut $(\Delta_M, \Delta_{\partial M})$ and get a partitioning scheme.
- Glue back with the Folding operation.



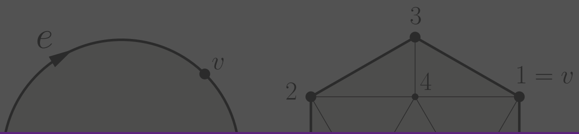
Real projective plane



The **projective plane** $\mathbb{R}P^2$ as a CW-complex and one of its triangulations.

Theorem (SG, 2020+)

Any triangulation of $\mathbb{R}P^2$ is partitionable.



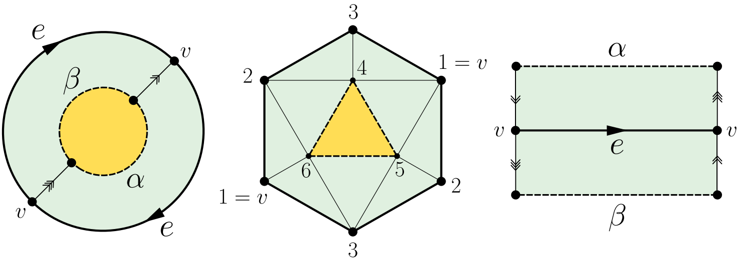
PROOF OUTLINE:

- Start with a simplicial triangulation $\Delta_{\mathbb{R}P^2}$ of $\mathbb{R}P^2$.
- Decompose $\Delta_{\mathbb{R}P^2}$ into two partitionable subcomplexes.
- Reconstruct $\Delta_{\mathbb{R}P^2}$ by applying the SHELLING-LIKE LEMMA.

THEOREM (SG, 2020+)

Any triangulation of $\mathbb{R}P^2$ is partitionable.

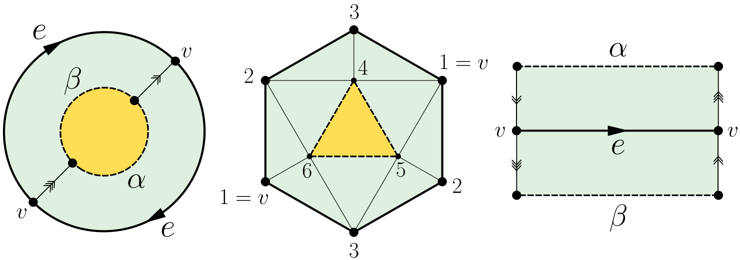
1 Decomposition



Decomposition of $\mathbb{RP}^2$ into a Möbius strip and a disk.

$\mathbb{R}P^2$ is partitionable

1 Decomposition



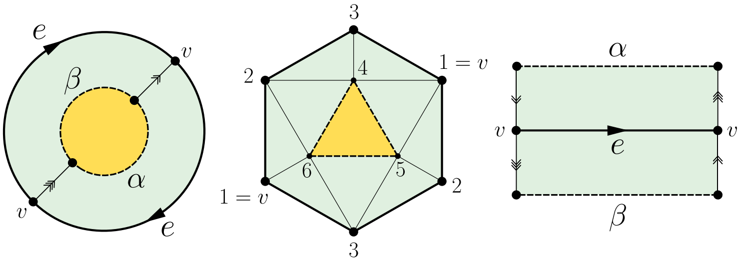
Decomposition of $\mathbb{R}P^2$ into a Möbius strip and a disk.

2 Reconstruction

$$\left.\begin{array}{r} \Delta_D \\ (\Delta_M, \Delta_D \cap \Delta_M) \end{array}\right\} \text{ partitionable}$$

$\mathbb{R}P^2$ is partitionable

1 Decomposition

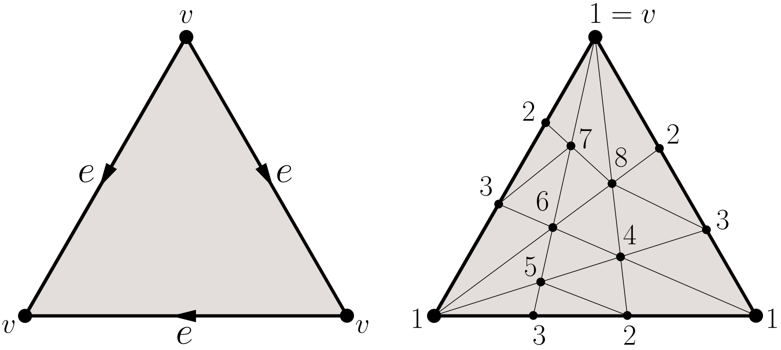


Decomposition of $\mathbb{R}P^2$ into a Möbius strip and a disk.

2 Reconstruction

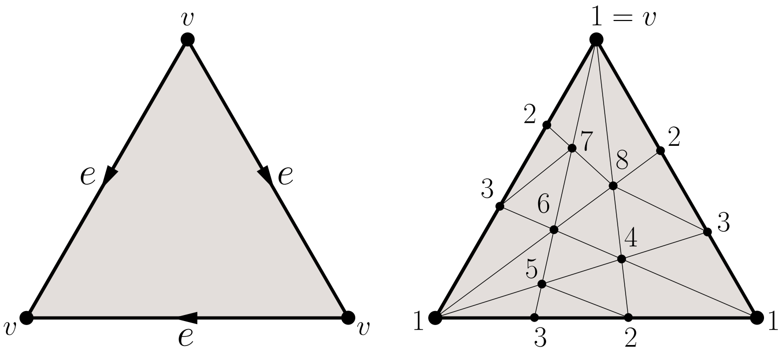
$$\left.\begin{matrix} \Delta_D \\ (\Delta_M, \Delta_D \cap \Delta_M) \end{matrix}\right\} \text{ partitionable } \xRightarrow[\text{Shelling-like Lemma}]{} \Delta_{\mathbb{R}P^2} \text{ partitionable,}$$

Dunce hat Z



The **dunce hat** Z as a CW-complex (ZEEMAN, 1964) and one of its triangulations.

Dunce hat Z

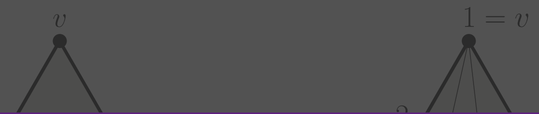


The **dunce hat** Z as a CW-complex (ZEEMAN, 1964) and one of its triangulations.

THEOREM (SG, 2020+)

Any triangulation of Z is partitionable.

Dunce hat Z



Proof outline:

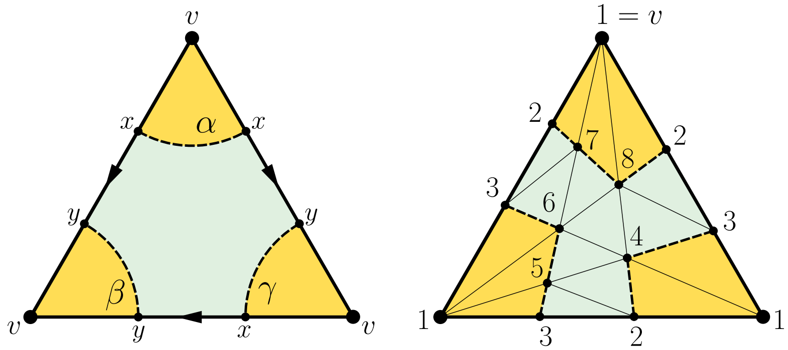
- Start with a simplicial triangulation Δ_Z of the dunce hat Z.
- Decompose Δ_Z into two complexes subcomplexes Λ and Σ.
- Shell Λ. Cut and fold Σ with the FOLDING OPERATION.
- Reconstruct Δ_Z by applying the SHELLING-LIKE LEMMA.

The **dunce hat** Z as a CW-complex (ZEEMAN, 1964) and one of its triangulations.

Theorem (SG, 2020+)

Any triangulation of Z is partitionable.

Z is partitionable



Decomposition of Z into the complexes Λ (yellow) and Σ (green).

Thank you