

👉 On system of split generalised mixed equilibrium and fixed point problems for multivalued mappings with no prior knowledge of operator norm.

Presented

by

O. T. Mewomo

8th European Congress of Mathematics  
20-26 June, 2021, Portorož, Slovenia

University of KwaZulu-Natal

## Fixed-point iterative schemes

O.T Mewomo

Some  
non-linear and  
optimization  
problems

Some notable  
fixed-point  
iterative  
schemes

Some  
iterative  
schemes for  
approximating  
solutions of  
nonlinear  
problems and  
fixed points of  
non-linear  
operators

References

- 1 Some non-linear and optimization problems
  - Introduction
  - Optimization problems
- 2 Some notable fixed-point iterative schemes
- 3 Some iterative schemes for approximating solutions of nonlinear problems and fixed points of non-linear operators
  - Recent developments
  - On system of split generalised mixed equilibrium and fixed point problems for multivalued mappings
- 4 References



Fixed-point  
iterative  
schemes

O.T Mewomo

Some  
non-linear and  
optimization  
problems

Some notable  
fixed-point  
iterative  
schemes

Some  
iterative  
schemes for  
approximating  
solutions of  
nonlinear  
problems and  
fixed points of  
non-linear  
operators

References

I acknowledge with thanks the invitation and opportunity by the organizers of the 8th European Congress of Mathematics.



## Fixed-point iterative schemes

O.T Mewomo

Some  
non-linear and  
optimization  
problems

Some notable  
fixed-point  
iterative  
schemes

Some  
iterative  
schemes for  
approximating  
solutions of  
nonlinear  
problems and  
fixed points of  
non-linear  
operators

References

In this presentation, we give a brief introduction to some notable fixed-point iterative methods and their convergence analysis in Hilbert spaces. We show how iterative methods can produce approximate solutions to certain classes of nonlinear and optimization problems, fixed points of some non-linear operators. Finally, we discuss one of our latest results and contributions.

- Optimization Problems (OPs) which includes minimization problems, variational inequality problems, equilibrium problems, among others, are known to be very useful in diverse fields such as ecology, physics, economics, computer science and engineering, since many problems arising from these fields can be modeled as an OP.
- One of the most successful and effective methods for solving OPs is the fixed point method. As a result of this, a lot of research efforts have been devoted in developing different fixed point iterative algorithms for approximating the solutions of OPs.
- In general, finding a solution of an optimization problem is equivalent to finding a fixed point for a suitable nonlinear mapping.

- The VIP is defined as finding a point  $x^* \in C$  such that

$$\langle Ax^*, y - x^* \rangle \geq 0, \quad \forall y \in E, \quad (1.1)$$

where  $A : C \rightarrow E^*$  is a nonlinear operator,  $E$  is a Banach space with dual  $E^*$  and  $C$  is nonempty, closed, convex subset of  $E$ .

- The first general theorem for the existence and uniqueness of solution of VIP was proved by Lions and Stampacchia [Li] in 1967. Since then, several authors have introduced various iterative methods for finding solutions of the VIP.

## Equilibrium Problem (EP)

- Another important optimization problem which has found many applications in solving real life problems is the EP introduced by Blum and Oettli [Bo] as a generalization of VIP.
- The EP is defined as finding a point  $x \in C$  such that

$$F(x, y) \geq 0, \quad \forall y \in C,$$

(1.2)

where  $F : C \times C \rightarrow \mathbb{R}$  is a bifunction.

- Blum and Oettli [Bo] established the existence of solutions of EP. Since then, various generalizations of EP and iterative methods for finding solutions of EP have been introduced and studied by many authors.

- The Generalized Mixed Equilibrium Problem (GMEP) is defined as finding a point  $x \in C$  such that

$$F(x, y) + \langle hx, y - x \rangle + \phi(y) - \phi(x) \geq 0, \quad \forall y \in C, \quad (1.3)$$

where  $h : C \rightarrow E$  is a nonlinear mapping and  $\phi : C \rightarrow \mathbb{R} \cup \{+\infty\}$  is a proper convex lower semicontinuous function.

- If  $h = 0$ , then GMEP reduces to the Mixed Equilibrium Problem (MEP) which is to find a point  $x \in C$  such that

$$F(x, y) + \phi(y) - \phi(x) \geq 0, \quad \forall y \in C. \quad (1.4)$$

- If  $\phi = 0$ , then GMEP reduces to a Generalized Equilibrium Problem (GEP) which is to find a point  $x \in C$  such that

$$F(x, y) + \langle hx, y - x \rangle \geq 0, \quad \forall y \in C. \quad (1.5)$$

- In particular, if  $h = 0$  and  $\phi = 0$ , the GMEP reduces to the EP.
- Several iterative methods for finding solutions of these generalizations of EP have been introduced by many authors.

- Let  $H_1, H_2$  be Hilbert spaces,  $C$  and  $D$  nonempty closed and convex subsets of  $H_1$  and  $H_2$ , respectively. Let  $f : C \times C \rightarrow \mathbb{R}$ ,  $g : D \times D \rightarrow \mathbb{R}$  be nonlinear bifunctions,  $P : C \rightarrow H_1$ ,  $Q : D \rightarrow H_2$ , be nonlinear mappings, and  $\phi : C \rightarrow \mathbb{R} \cup \{+\infty\}$ ,  $\varphi : D \rightarrow \mathbb{R} \cup \{+\infty\}$  be proper lower semicontinuous and convex functions. Let  $A : H_1 \rightarrow H_2$  be a bounded linear operator. The Split Generalised Mixed Equilibrium Problem (SGMEP) is to find a point  $\hat{x} \in C$  such that

$$f(\hat{x}, x) + \langle P\hat{x}, x - \hat{x} \rangle + \phi(x) - \phi(\hat{x}) \geq 0, \quad \text{for all } x \in C, \quad (1.6)$$

and  $\hat{y} = A\hat{x} \in D$  solves

$$g(\hat{y}, y) + \langle Q\hat{y}, y - \hat{y} \rangle + \varphi(y) - \varphi(\hat{y}) \geq 0, \quad \text{for all } y \in D. \quad (1.7)$$

### Fixed-point iterative schemes

O.T Mewomo

### Some non-linear and optimization problems

Introduction

Optimization  
problems

### Some notable fixed-point iterative schemes

### Some iterative schemes for approximating solutions of nonlinear problems and fixed points of non-linear operators

References

- We denote the solution set of (1.6)-(1.7) by  $\Gamma = \{\hat{x} \in GMEP(f, P, \phi) : A\hat{x} \in GMEP(g, Q, \varphi)\}$ .
- If  $P = Q = 0$ , then (1.6)-(1.7) reduces to the Split Mixed Equilibrium Problem (SMEP) introduced by Onjai-uea and Phuengrattana 2017.
- Also, if  $\phi = \varphi = 0$  in (1.6)-(1.7), we have the Split Generalised Equilibrium Problem (SGEP).
- Furthermore, if  $P = Q = 0$  and  $\phi = \varphi = 0$ , then the SGMEP (1.6)-(1.7) reduces to the Split Equilibrium Problem (SEP).

- A point  $x \in E$  is called a fixed point of the mapping  $T : E \rightarrow E$  if

$$Tx = x. \quad (1.8)$$

- When  $T$  is a multi-valued mapping, e.g.  $T : E \rightarrow 2^E$ , then a point  $x \in E$  is called a fixed point of  $T$  if  $x \in Tx$ .
- Denote the set of fixed points of  $T$  by denoted by  $F(T)$ .
- The fixed point theory for multi-valued mappings can be utilized in various areas such as game theory, control theory, mathematical economics, etc.
- In general, finding a solution of an optimization problem is equivalent to solving FPP, i.e. finding a fixed point for a suitable nonlinear mapping.

# Banach Contraction Mapping Principle (BCMP)

## Picard Iteration Process (PIP)

Fixed-point  
iterative  
schemes

O.T Mewomo

Some  
non-linear and  
optimization  
problems

Some notable  
fixed-point  
iterative  
schemes

Some  
iterative  
schemes for  
approximating  
solutions of  
nonlinear  
problems and  
fixed points of  
non-linear  
operators

References

- PIP is defined in a metric space  $X$  as follows:

$$\begin{cases} x_1 \in X, \\ x_{n+1} = Tx_n, \quad \forall n \geq 1, \end{cases} \quad (2.1)$$

- BCMP (see [Ba]): For a complete metric space  $X$  and  $T : X \rightarrow X$  a contraction. Then
  - $T$  has a unique fixed point
  - PIP (3.1) converges strongly to the unique fixed point of  $T$ .
- BCMP is the pivot of metric fixed point theory.
- For mappings more general than the contraction mapping, one may not be able to apply the BCMP.

# Krasnoselskii Iteration Process (KIP)

Fixed-point  
iterative  
schemes

O.T Mewomo

Some  
non-linear and  
optimization  
problems

Some notable  
fixed-point  
iterative  
schemes

Some  
iterative  
schemes for  
approximating  
solutions of  
nonlinear  
problems and  
fixed points of  
non-linear  
operators

References

- Take  $C$  a closed subset of  $X$ .

$$\begin{cases} x_1 \in X, \\ x_{n+1} = \frac{1}{2}(x_n + Tx_n), \quad \forall n \geq 1, \end{cases} \quad (2.2)$$

is a generalization of PIP (2.1) by replacing  $T$  in (2.1) by  $\frac{1}{2}(I + T)$ .

- Clearly  $T$  and  $\frac{1}{2}(I + T)$  have the same set of fixed points. So, the limit of sequence (2.2) is the fixed point of  $T$ .
- For  $X$  a normed space and  $T$  nonexpansive. The KIP is a generalization of (2.2) and is defined as follows:

$$\begin{cases} x_1 \in X, \\ x_{n+1} = (1 - \lambda)x_n + \lambda Tx_n, \quad \forall n \geq 1. \end{cases} \quad (2.3)$$

- KIP is exactly the PIP corresponding to the average operator  $T_\lambda = (1 - \lambda)I + \lambda T$ , i.e. replace  $T$  with  $(1 - \lambda)I + \lambda T$  in PIP to get KIP

- MIP introduced by Mann [Ma], is defined in a real Hilbert space  $H$  as follows:

$$\begin{cases} x_1 \in H, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \quad \forall n \geq 1, \end{cases} \quad (2.4)$$

where  $\{\alpha_n\}$  is a sequence in  $[0, 1]$ .

- If  $T_n = (1 - \alpha_n)I + \alpha_n T$ , then  $F(T) = F(T_n)$  for  $\{\alpha_n\} \subset [0, 1]$ .
- If the sequence  $\alpha_n = \lambda$  (constant), then MIP reduces to KIP.
- MIP has only weak convergence in general.
- It is well known that, if the MIP converges, then it will converge to a fixed point of a continuous mapping  $T$ .
- However, if  $T$  is not continuous, then the MIP may fail to converge to a fixed point of  $T$  even when it converges

(see [BHE, C, KRA])

## Ishikawa Iteration Process (IIP)

Fixed-point  
iterative  
schemes

O.T Mewomo

Some  
non-linear and  
optimization  
problems

Some notable  
fixed-point  
iterative  
schemes

Some  
iterative  
schemes for  
approximating  
solutions of  
nonlinear  
problems and  
fixed points of  
non-linear  
operators

References

- In 1974, Ishikawa [Ish] introduced the following generalization of MIP, called the IIP for approximating fixed points of pseudocontractive mappings in Hilbert spaces.

- 

$$\begin{cases} x_1 \in H, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n, \\ y_n = (1 - \beta_n)x_n + \beta_n T x_n, \quad \forall n \geq 1, \end{cases} \quad (2.5)$$

where  $\{\alpha_n\}$  and  $\{\beta_n\}$  are sequences in  $[0, 1]$ .

- If  $\beta_n = 0$ , then IIP reduces to MIP.

## S-Iteration Process (SIP)

Fixed-point  
iterative  
schemes

O.T Mewomo

Some  
non-linear and  
optimization  
problems

Some notable  
fixed-point  
iterative  
schemes

Some  
iterative  
schemes for  
approximating  
solutions of  
nonlinear  
problems and  
fixed points of  
non-linear  
operators

References

- In (2007), Agarwal *et al.* [Arg] introduced and studied the following SIP:



$$\begin{cases} x_1 \in H, \\ x_{n+1} = (1 - \alpha_n)Tx_n + \alpha_nTy_n \\ y_n = (1 - \beta_n)x_n + \beta_nTx_n \quad \forall n \geq 1, \end{cases} \quad (2.6)$$

where  $\{\alpha_n\}$  and  $\{\beta_n\}$  are sequences in  $(0, 1)$ .

- SIP is independent of MIP and IIS, and has better convergence rate than MIP and ISI, see [Arg] .

- In general, the Picard, Mann, Ishikawa and  $S$ -iteration processes only converge weakly.
- However, in infinite dimensional spaces, strong convergence are more desirable and interesting than weak convergence.
- For this reason (among others), Halpern [BHF] introduced the following HIP which converges strongly to a fixed point of a nonexpansive mapping in real Hilbert spaces.

•

$$\begin{cases} u, x_1 \in H, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad \forall n \geq 1, \end{cases} \quad (2.7)$$

where  $\{\alpha_n\}$  is a sequence in  $[0, 1]$ .

## Viscosity Iterative Process (VIP)

Fixed-point  
iterative  
schemes

O.T Mewomo

Some  
non-linear and  
optimization  
problems

Some notable  
fixed-point  
iterative  
schemes

Some  
iterative  
schemes for  
approximating  
solutions of  
nonlinear  
problems and  
fixed points of  
non-linear  
operators

References

- An important generalization of the HIP is the VIP in real Hilbert spaces by XU [Xu], as follows:

- 

$$\begin{cases} x_1 \in H, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \quad \forall n \geq 1, \end{cases} \quad (2.8)$$

where  $\{\alpha_n\}$  is a sequence in  $[0, 1]$  and  $f$  is a contractive mapping on  $X$ .

- VIP converges strongly to a unique solution of some variational inequalities associated with  $f$  and have higher rate of convergence than the HIP (see [OIM,YSX]).

# Recent developments

## Contribution by several authors

Fixed-point  
iterative  
schemes

O.T Mewomo

Some  
non-linear and  
optimization  
problems

Some notable  
fixed-point  
iterative  
schemes

Some  
iterative  
schemes for  
approximating  
solutions of  
nonlinear  
problems and  
fixed points of  
non-linear  
operators

Recent  
developments

On system of split  
generalised mixed  
equilibrium and fixed

- In the recent time, several authors have studied the above iteration processes and their modifications to approximate fixed points of nonlinear mappings and solutions of optimization problems in Hilbert, Banach and other important spaces.
- This is an active area of research with numerous research articles in literature mainly published in
  - Journal of Fixed Point Theory and Applications
  - Fixed Point Theory
  - Fixed Point Theory and Applications
  - Optimization
  - Numerical Algorithms
  - Journal of Optimization Theory and Applications
  - Numerical Functional Analysis and Applications
  - Journal of Nonlinear and Convex Analysis
  - and so on.

# One of our latest results

## Accepted, Fixed Point Theory (2021)

Fixed-point  
iterative  
schemes

O.T Mewomo

Some  
non-linear and  
optimization  
problems

Some notable  
fixed-point  
iterative  
schemes

Some  
iterative  
schemes for  
approximating  
solutions of  
nonlinear  
problems and  
fixed points of  
non-linear  
operators

Recent  
developments

On system of split  
generalised mixed  
equilibrium and fixed

- We introduce the notion of System of Split Generalised Mixed Equilibrium Problem (SSGMEP).
- We propose a new iterative algorithm of inertial form to find a common solution of the (SSGMEP) and Fixed Point Problem (FPP) in real Hilbert spaces.
- We incorporate self adaptive step size and inertial technique to overcome the difficulty of having to compute the operator norm and to accelerate the convergence of the proposed method.
- We prove that the proposed method converges strongly to a common solution of the SSGMEP and fixed point of finite family of multivalued demicontractive mappings.
- Finally, we give some numerical experiments to illustrate the performance of our method.

- Let  $C_i$  and  $D_i$  be nonempty closed convex subsets of  $H_1$  and  $H_2$ , respectively,  $i = 1 \leq i \leq m$ . Let  $A_i : H_1 \rightarrow H_2$  be bounded linear operators,  $f_i : C_i \times C_i \rightarrow \mathbb{R}$  and  $g_i : D_i \times D_i \rightarrow \mathbb{R}$ , nonlinear bifunctions,  $P_i : C_i \rightarrow H_1$ ,  $Q_i : D_i \rightarrow H_2$ , nonlinear mappings, and let  $\phi_i : C_i \rightarrow \mathbb{R} \cup \{+\infty\}$ ,  $\varphi_i : D_i \rightarrow \mathbb{R} \cup \{+\infty\}$  be proper lower semicontinuous and convex functions such that  $\bigcap_{i=1}^m C_i \neq \emptyset$  and  $\bigcap_{i=1}^m D_i \neq \emptyset$ . The SSGMEP is to find  $\hat{x} \in C = \bigcap_{i=1}^m C_i$  such that

$$f_i(\hat{x}, x) + \langle P_i \hat{x}, x - \hat{x} \rangle + \phi_i(x) - \phi_i(\hat{x}) \geq 0, \quad \text{for all } x \in C_i, \quad (3.1)$$

and  $\hat{y} = A_i \hat{x} \in D = \bigcap_{i=1}^m D_i$  solves

$$g_i(\hat{y}, y) + \langle Q_i \hat{y}, y - \hat{y} \rangle + \varphi_i(y) - \varphi_i(\hat{y}) \geq 0, \quad \text{for all } y \in D_i. \quad (3.2)$$

- We denote the solution set of (3.1)-(3.2) by  $\Omega = \{\hat{x} \in \bigcap_{i=1}^m GMEP(f_i, P_i, \phi_i) : A_i \hat{x} \in \bigcap_{i=1}^m GMEP(g_i, Q_i, \varphi_i)\}$ .
- If  $P_i = Q_i = 0$ , then (3.1)-(3.2) reduces to the System of Split Mixed Equilibrium Problem (SSMEP) introduced by Karahan in 2019.
- Also, if  $\phi_i = \varphi_i = 0$  in (3.1)-(3.2), we have the System of Split Generalised Equilibrium Problem (SSGEP).
- Furthermore, if  $P_i = Q_i = 0$  and  $\phi_i = \varphi_i = 0$ , then the (SSGMEP) (3.1)-(3.2) reduces to the System of Split Equilibrium Problem (SSEP) introduced by Ugwunnadi and Ali in (2016).
- Observe that if  $m = 1$ , the new problem introduced reduces to the SGMEP (1.6)-(1.7). Hence, our new problem is a generalization of SGMEP.

- More precisely, we consider the following problem: Find  $x^* \in \bigcap_{i=1}^m F(S_i)$ , such that

$$f_i(x^*, x) + \langle P_i x^*, x - x^* \rangle + \phi_i(x) - \phi_i(x^*) \geq 0, \text{ for all } x \in C_i \quad (3.3)$$

and  $\hat{y} = A_i \hat{x} \in D = \bigcap_{i=1}^m D_i$  solves

$$g_i(y^*, y) + \langle Q_i y^*, y - y^* \rangle + \psi_i(y) - \psi_i(y^*) \geq 0, \text{ for all } y \in D_i, \quad (3.4)$$

where  $S_i : C_i \rightarrow CB(C_i)$  is a finite family of multivalued demicontractive mappings.

- $CB(C)$  family nonempty closed bounded subsets of  $C$ .
- The Pompeiu-Hausdorff metric on  $CB(C)$  is defined by

$$H(A, B) := \max\left\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\right\},$$

for all  $A, B \in CB(C)$ .

- $S : C \rightarrow 2^C$  a multivalued mapping. We say that  $S$  satisfies the endpoint condition if  $S_p = \{p\}$  for all  $p \in F(S)$ .
- For multivalued mappings  $S_i : C \rightarrow 2^C (i \in \mathbb{N})$  with  $\bigcap_{i=1}^{\infty} F(S_i) \neq \emptyset$ , we say  $S_i$  satisfies the common endpoint condition if  $S_i(p) = \{p\}$  for all  $i \in \mathbb{N}, p \in \bigcap_{i=1}^{\infty} F(S_i)$ .
- A multivalued mapping  $S : C \rightarrow CB(C)$  is said to be  $k$ -demicontractive for  $0 \leq k < 1$  if  $F(S) \neq \emptyset$  and
 
$$H(Sx, Sp)^2 \leq \|x-p\|^2 + kd(x, Sx)^2, \text{ for all } x \in C, p \in F(S).$$

- $C_i, D_i$  are nonempty closed convex subsets  $H_1, H_2$  respectively,  $A_i : H_1 \rightarrow H_2$  a finite family of bounded linear operators with adjoint  $A_i^*$ ,  $f_i : C_i \times C_i \rightarrow \mathbb{R}$  and  $g_i : D_i \times D_i \rightarrow \mathbb{R}$  are two finite families of bifunctions satisfying conditions (A1) - (A4).
- The bifunction  $f$  is assumed to satisfy the following conditions:
  - (A1)  $f(x, x) = 0$  for all  $x \in C$ ;
  - (A2)  $f$  is monotone, i.e.  $f(x, y) + f(y, x) \leq 0$  for all  $x, y \in C$ ;
  - (A3) for each  $x, y, z \in C$ ,  $\lim_{t \rightarrow 0} f(tz + (1-t)x, y) \leq f(x, y)$ ;
  - (A4) for each  $x \in C, y \rightarrow f(x, y)$  is convex and lower semicontinuous.

- Let  $P : C \rightarrow H$  be a continuous and monotone mapping,  $\phi : C \rightarrow \mathbb{R} \cup \{+\infty\}$  a proper lower semicontinuous and convex function, and  $f : C \times C \rightarrow \mathbb{R}$  be a bifunction that satisfies (A1) – (A4). For  $r > 0$  and  $x \in H$ , there exists  $u \in C$  such that

$$f(u, y) + \langle Pu, y - u \rangle + \phi(y) - \phi(u) + \frac{1}{r} \langle y - u, u - x \rangle \geq 0, \forall y \in C.$$

- Define a resolvent function  $T_r^f : H \rightarrow C$  as follows:  
 $T_r^f(x) = \{u \in C : f(u, y) + \langle Pu, y - u \rangle + \phi(y) - \phi(u) + \frac{1}{r} \langle y - u, u - x \rangle \geq 0, \forall y \in C\}.$

Then the following conclusions hold:

- for each  $x \in H$ ,  $T_r^f(x) \neq \emptyset$ ,
- $T_r^f$  is single-valued,
- $T_r^f$  is firmly nonexpansive, i.e. for any  $x, y \in H$ ,  $\|T_r^f(x) - T_r^f(y)\|^2 \leq \langle T_r^f(x) - T_r^f(y), x - y \rangle$ ,
- $F(T_r^f) = GMEP(f, P, \phi)$ ,
- $GMEP(f, P, \phi)$  is closed and convex.

## Assumptions on the control parameters

- (B1)  $\{\beta_{n,i}\}, \{\delta_{n,i}\} \subset (0, 1), \sum_{i=0}^m \beta_{n,i} = \sum_{i=0}^m \delta_{n,i} = 1;$
- (B2)  $\liminf_n \beta_{n,0} \beta_{n,i} > 0,$  and  $\liminf_n (\delta_{n,0} - k) \delta_{n,i} > 0,$  for each  $1 \leq i \leq m;$
- (B3)  $\{\alpha_n\}, \{\xi_n\}$  and  $\{\gamma_n\}$  are real sequences in  $(0, 1)$  such that  $\alpha_n + \xi_n + \gamma_n = 1;$
- (B4)  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=0}^{\infty} \alpha_n = \infty, 0 < c_1 \leq \xi_n, 0 < c_2 \leq \gamma_n, 0 < a \leq \tau_n \leq b < 1;$
- (B5)  $\{r_{n,i}\}, \{s_{n,i}\}$  are positive real sequences such that  $\liminf_{n \rightarrow \infty} r_{n,i} > 0$  and  $\liminf_{n \rightarrow \infty} s_{n,i} > 0;$
- (B6) Let  $\theta \geq 3$  and let  $\{\epsilon_n\}$  be nonnegative sequence such that  $0 < d \leq \epsilon_n;$
- (B7)  $\epsilon_n = o(\alpha_n),$  i.e.,  $\lim_{n \rightarrow \infty} \frac{\epsilon_n}{\alpha_n} = 0,$  (e.g.  $\epsilon_n = \frac{1}{(n+1)^2}, \alpha_n = \frac{1}{n+1}$ ).

Fixed-point  
iterative  
schemes

O.T Mewomo

Some  
non-linear and  
optimization  
problems

Some notable  
fixed-point  
iterative  
schemes

Some  
iterative  
schemes for  
approximating  
solutions of  
nonlinear  
problems and  
fixed points of  
non-linear  
operators

Recent  
developments

On system of split  
generalised mixed  
equilibrium and fixed

## Our Proposed Method (Algorithm)

**Step 0.** Select initial data  $x_0, x_1 \in C$  and set  $n = 1$ .

**Step 1.** Given the  $(n - 1)$ th and  $n$ th iterates, choose  $\theta_n$  such that  $0 \leq \theta_n \leq \hat{\theta}_n$  with  $\hat{\theta}_n$  defined by

$$\hat{\theta}_n = \begin{cases} \min \left\{ \frac{n-1}{n+\theta-1}, \frac{\epsilon_n}{\|x_n - x_{n-1}\|} \right\}, & \text{if } x_n \neq x_{n-1}, \\ \frac{n-1}{n+\theta-1}, & \text{otherwise.} \end{cases} \quad (3.5)$$

**Step 2.** Compute

$$w_n = x_n + \theta_n(x_n - x_{n-1}). \quad (3.6)$$

**Step 3.** Compute

$$z_{n,i} = w_n - \lambda_{n,i} A_i^* (A_i w_n - T_{s_{n,i}}^{g_i} A_i w_n), \quad (3.7)$$

where

$$\lambda_{n,i} := \begin{cases} \tau_n \frac{\|(I - T_{s_{n,i}}^{g_i})A_i w_n\|^2}{\|A_i^*(I - T_{s_{n,i}}^{g_i})A_i w_n\|^2}, & \text{if } A_i w_n \neq T_{s_{n,i}}^{g_i} A_i w_n, \\ \lambda, & \text{otherwise } (\lambda \text{ being any nonnegative real number}). \end{cases} \quad (3.8)$$

Step 4. Compute

$$\begin{cases} u_n = \beta_{n,0} w_n + \sum_{i=1}^m \beta_{n,i} T_{r_{n,i}}^{f_i} z_{n,i} \\ y_n = \delta_{n,0} u_n + \sum_{i=1}^m \delta_{n,i} v_{n,i} \\ x_{n+1} = \alpha_n h_n(x_n) + \xi_n x_n + \gamma_n y_n, \end{cases} \quad (3.9)$$

where  $v_{n,i} \in S_i u_n$ . Set  $n := n + 1$  and return to **Step 1**.

- The notable advantages of our algorithm for solving SSGMEP are that the step size is self adaptive and inertial technique is employed to accelerate the convergence of the proposed method.
- The choice of the stepsize  $\lambda_{n,i}$  in (3.8) used in our algorithm does not require priori knowledge of the operator norm  $\|A_i\|$ . Algorithms with stepsize that depends on the operator norm require the computation of the norm of the bounded linear operator, which in general is a very difficult task (sometimes impossible) to accomplish. Moreover, the step size defined by such algorithms are often very small and deteriorates the convergence rate of the algorithm. In practice, a larger step size can often be used to yield better numerical results.

## Lemma

*The sequence  $\{x_n\}$  generated by our proposed algorithm is bounded.*

## Theorem

*The sequence  $\{x_n\}$  generated by our proposed algorithm such that Assumptions (A1)-(A4) and (B1) - (B7) are satisfied converges strongly to a point  $\hat{x} \in \Omega \cap \bigcap_{i=1}^m F(S_i)$ , where  $\hat{x} = P_{\Omega \cap \bigcap_{i=1}^m F(S_i)} h(\hat{x})$ .*

- See Fixed Point Theory, Volume 22, No. 2, (2021), (to appear), for proof of the above results, further results and numerical experiments.

Fixed-point  
iterative  
schemes

O.T Mewomo

Some  
non-linear and  
optimization  
problems

Some notable  
fixed-point  
iterative  
schemes

Some  
iterative  
schemes for  
approximating  
solutions of  
nonlinear  
problems and  
fixed points of  
non-linear  
operators

Recent  
developments

On system of split  
generalised mixed  
equilibrium and fixed

● **THANKS FOR LISTEN**

- Arg** R. P. Agarwal, D. O'Regan, D. R. Sahu, Iterative construction of fixed points of nearly asymptotically nonexpansive mappings, *J. Nonlinear Convex Anal.*, **8** (2007), 61-79.
- Ba** S. Banach, Sur les operations dans les ensembles abstracts ET leur applications aux equations integrals, *Fund. Math.*, **3** (1922), 133-181.
- Bo** E. Blum and W. Oettli, From optimization and variational inequalities to equilibrium problems, *Math. Stud.*, **63** (1994), 123-145.
- Ce** Y. Censor, T. Elfving, A multiprojection algorithm using Bregman projections in a product space, *Numerical Algorithms*, **8**(2)(1994), 221 - 239.
- CaY** L. C. Ceng, Q. H. Ansari and J. C. Yao, An extragradient method for solving split feasibility and fixed point problems, *Comput. Math. Appl.* **64** (2012)

- C** C. E. Chidume, Geometric properties of Banach spaces and nonlinear iterations, *Springer Verlag Series, Lecture Notes in Mathematics*, ISBN 978-1-84882-189-7, (2009).
- G** O. Güler, On the convergence of the proximal point algorithm for convex minimization, *SIAM J. Control Optim.*, **29** (1991), 403-419.
- BHF** B. Halpern, Fixed points of nonexpanding maps, *Bull. Amer. Math. Soc.*, **73** (1967), 591-597.
- Ish** S. Ishikawa, Fixed points by a new iteration method, *Proc. Amer. Math. Soc.*, **44** (1) (1974), 147-150.
- KT** S. Kamimura and W. Takahashi, Approximating solutions of maximal monotone operators in Hilbert spaces, *J. Approx. Theory*, **106** (2000), 226-240.

- KRA** M. A. Krasnosel'skii, Two observations about the method of successive approximations, *Uspehi. Math. Nauk.*, 10(1955), 123-127.
- Li** J.L. Lions and G. Stampacchia, Variational inequalities, *COmm. Pure Appl. Math.*, **20** (1967), 493-519.
- Ma** W. R. Mann, Mean value methods in iteration, *Proc. Amer. Math. Soc.*, **4** (1953), 506-510.
- OIM** C.C. Okeke, C. Izuchukwu, O.T. Mewomo, Strong convergence results for convex minimization and monotone variational inclusion problems in Hilbert space, *Rendiconti del Circolo Matematico di Palermo Series 2* (2019). <https://doi.org/10.1007/s12215-019-00427-y>.
- Roc** R. T. Rockafellar, Monotone operators and the proximal point algorithm, *SIAM J. Control Optim.*, **14** (1976), 877-898.

- YSX** Y. Song and X. Liu, Convergence comparison of several iteration algorithms for the common fixed point problems, *Fixed Point Theory Appl.*, **2009** (2009), Art. ID 824374.
- Xu** H. K. Xu, Viscosity approximation methods for nonexpansive mapping, *J. Math. Anal. Appl.*, **298** (1) (2004), 279-291.

Fixed-point  
iterative  
schemes

O.T Mewomo

Some  
non-linear and  
optimization  
problems

Some notable  
fixed-point  
iterative  
schemes

Some  
iterative  
schemes for  
approximating  
solutions of  
nonlinear  
problems and  
fixed points of  
non-linear  
operators

References

**Thank You!**