

Nonstandard finite elements for wave problems

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Joint work with

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PDE problem - variational formulation

$$\mathcal{L}u = f \quad \text{in } \Omega \quad \text{with boundary conditions}$$

$$\text{find } u \in V: \quad a(u, v) = \ell(v) \quad \forall v \in V$$

Galerkin method

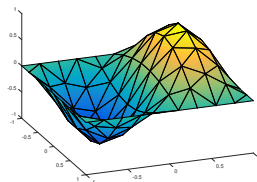
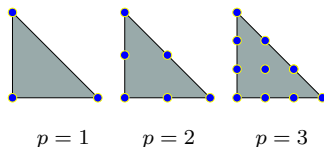
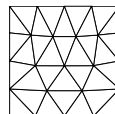
$$\text{find } u_N \in V_N \subset V: \quad a(u_N, v_N) = \ell(v_N) \quad \forall v_N \in V_N, \quad \dim(V_N) = N < +\infty$$

$$A\mathbf{u} = \mathbf{b} \quad N \times N \text{ algebraic linear system}$$

- $V_N \subset V$ (conformity)
- same $a(\cdot, \cdot)$ and $\ell(\cdot)$ as for the continuous problem

choice of V_N

- partition \mathcal{T}_h of Ω into simplices or bricks
- $V_p(\mathcal{T}_h)$ = space of **piecewise polynomials** of degree p on \mathcal{T}_h such that $V_p(\mathcal{T}_h) \subset V$ (conformity)

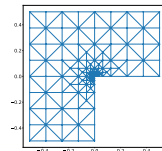
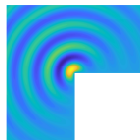
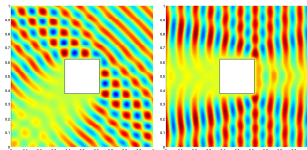


time-dependent problems

- finite elements in space \rightarrow system of ODEs e.g. $\dot{\mathbf{u}} + \mathbf{A}\mathbf{u}(t) = \mathbf{b}(t)$
- ODE solver (time-stepping)

- highly oscillatory solutions → high resolution is required (high p in space convenient)
- multimaterials → different resolutions in different subdomains
- solution singularities → locally refined spatial meshes required
- the presence of small spatial elements forces small time-steps to avoid instabilities
 - CFL condition

nonstandard / *ad hoc* approaches ?



standard

- variational formulation in space and time-stepping
- mesh in space \times partition of the time interval
- Galerkin setting: $V_N \subset V$, $a_N(\cdot, \cdot) = a(\cdot, \cdot)$, $\ell_N(\cdot) = \ell(\cdot)$
- local basis functions in complete polynomial spaces

nonstandard

- space-time variational formulation
- subcharacteristic, tent-shaped meshes of the space-time domain
- discontinuous Galerkin setting: $V_N \not\subset V$, $a_N(\cdot, \cdot) \neq a(\cdot, \cdot)$, $\ell_N(\cdot) \neq \ell(\cdot)$
- local basis functions chosen depending on the PDE operator

why space-time?

- *high-order* approximation in both space and time is simple to obtain
- stability is achieved under a *local* CFL condition
- the numerical solution is available at *all times* in $(0, T)$

drawback: high complexity

time dependent problem in d space dimensions $\rightarrow (d + 1)$ -dimensional problem

- model problem: the acoustic wave equation
- space-time discontinuous Galerkin (DG) discretization
- reduction of the complexity:
 - special (Trefftz) basis functions and tent-pitched meshes [1], [2]
 - tensor-product (in time) elements and combination formula [3]



[1] A. Moiola, I. Perugia, A space-time Trefftz discontinuous Galerkin method for the acoustic wave equation in first-order formulation, *Num. Math.*, 139 (2018), 389-435.



[2] I. Perugia, J. Schöberl, P. Stocker, C. Wintersteiger, Tent pitching and Trefftz-DG method for the acoustic wave equation, *Comput. Math. with Appl.*, 70 (2020), 2987-3000.



[3] P. Bansal, A. Moiola, I. Perugia, C. Schwab, Space-time discontinuous Galerkin approximation of acoustic waves with point singularities, *IMA J. Numer. Anal.*, online.

space-time finite element methods for wave problems

- early works (FEM): [Hughes, Hulbert, 1988, 1990], [French, 1993], [Johnson, 1993], ...
- DG: [Falk, Richter, 1999], [Yin, Acharya, Sobh, Haber, Tortorelli, 2000], [Monk, Richter, 2005], [Costanzo, Huang, 2005], [Abedi, Petracovici, Haber, 2006], [van der Vegt, 2006], [Feistauer, Hájek, Švadlenka, 2007], ..., [Gopalakrishnan, Monk, Sepúlveda, 2015], [Dörfler, Findeisen, Wieners, 2016], [Gopalakrishnan, Schöberl, Wintersteiger, 2017, 2019], ...
- Trefftz: [Maciąg, Wauer, Sokala, 2005–2011], [Liu, Kuo, 2016], [Petersen, Farhat, Tezaur, 2009], [Wang, Tezaur, Farhat, 2014], [Egger, Kretschmar, Schnepf, Tzukermann, Weiland, 2014, 2015], [Banjai, Georgoulis, Lijoka, 2017], [Barucq, Calandra, Diaz, Shishenina, 2018, 2020], [1], [2]
- recent, on tensor-product meshes: [Steinbach, Zank, 2019], [Ernesti, Wieners, 2019], [3]

the acoustic wave problem as a 1st order system

$Q = \Omega \times (0, T)$, $\Omega \subset \mathbb{R}^d$ Lipschitz, bounded polygon/polyhedron

$c = c(\mathbf{x})$ piecewise constant on a fixed, finite polygonal/polyhedral partition $\{\Omega_i\}$ of Ω

$f \in L^2(Q)$, $v_0 \in L^2(\Omega)$, $\boldsymbol{\sigma}_0 \in L^2(\Omega)^d$

$$\left\{ \begin{array}{ll} \text{find } (v, \boldsymbol{\sigma}) \text{ such that} & \\ \nabla v + \frac{\partial \boldsymbol{\sigma}}{\partial t} = \mathbf{0}, \quad \nabla \cdot \boldsymbol{\sigma} + c^{-2} \frac{\partial v}{\partial t} = f & \text{in } Q \\ v(\cdot, 0) = v_0, \quad \boldsymbol{\sigma}(\cdot, 0) = \boldsymbol{\sigma}_0 & \text{on } \Omega \\ v = 0 & \text{on } \partial\Omega \times [0, T] \end{array} \right.$$

2nd order wave equation (provided that $\boldsymbol{\sigma}_0$ is a gradient)

$$\begin{array}{l} v = \partial_t U \\ \boldsymbol{\sigma} = -\nabla U \end{array} \quad \rightarrow \quad \boxed{-\Delta U + c^{-2} \partial_{tt} U = f \quad \text{in } \Omega} \quad \begin{array}{l} + \text{ initial/boundary} \\ \text{conditions} \end{array}$$

$$U \in C^0([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega)) \cap H^2(0, T; H^{-1}(\Omega))$$

[Dautray, Lions, 1992]

$$\nabla v + \frac{\partial \boldsymbol{\sigma}}{\partial t} = \mathbf{0}, \quad \nabla \cdot \boldsymbol{\sigma} + c^{-2} \frac{\partial v}{\partial t} = f \quad \text{in } Q \quad \boxed{\mathcal{L}_{\text{wave}}(v, \boldsymbol{\sigma}) = (f, \mathbf{0})}$$

multiply by test functions $\boldsymbol{\tau}$ and w , respectively, and integrate by parts in $Q = \Omega \times (0, T)$:

$$\int_Q \left(\nabla v + \frac{\partial \boldsymbol{\sigma}}{\partial t} \right) \cdot \boldsymbol{\tau} \, dV + \int_Q \left(\nabla \cdot \boldsymbol{\sigma} + c^{-2} \frac{\partial v}{\partial t} \right) w \, dV = \int_Q f w \, dV \quad \curvearrowright$$

space-time variational formulation

$$\begin{aligned} - \int_Q \left[v \left(\nabla \cdot \boldsymbol{\tau} + c^{-2} \frac{\partial w}{\partial t} \right) + \boldsymbol{\sigma} \cdot \left(\nabla w + \frac{\partial \boldsymbol{\tau}}{\partial t} \right) \right] dV + \int_{\Omega \times \{T\}} (\boldsymbol{\sigma} \cdot \boldsymbol{\tau} + c^{-2} v w) \, dx \\ = \int_Q f w \, dV + \int_{\Omega \times \{0\}} (\boldsymbol{\sigma}_0 \cdot \boldsymbol{\tau} + c^{-2} v_0 w) \, dx \end{aligned}$$

- $\nabla v + \frac{\partial \boldsymbol{\sigma}}{\partial t} = \mathbf{0}$ holds in $C^0([0, T]; H_0(\text{div}; \Omega)^*)$
- $\nabla \cdot \boldsymbol{\sigma} + c^{-2} \frac{\partial v}{\partial t} = f$ holds in $L^2(0, T; H^{-1}(\Omega))$
- $v = 0$ on $\partial\Omega \times [0, T]$ is imposed weakly

(details in [3])

$$\nabla v + \frac{\partial \boldsymbol{\sigma}}{\partial t} = \mathbf{0}, \quad \nabla \cdot \boldsymbol{\sigma} + c^{-2} \frac{\partial v}{\partial t} = f \quad \text{in } Q = \Omega \times (0, T)$$

- introduce a polytopic **space-time mesh** $\mathcal{T}_h = \{K\}$ of Q , with c constant in each element
- multiply by test functions and integrate by parts **element by element**
- discretize $(v, \boldsymbol{\sigma})$ and $(w, \boldsymbol{\tau})$ in **discontinuous, piecewise polynomial spaces** $\mathbf{V}_p(\mathcal{T}_h)$
- replace interelement traces by **numerical fluxes**

elemental DG formulation

$$\begin{aligned}
 & - \int_K \left[v_h \left(\nabla \cdot \boldsymbol{\tau}_h + c^{-2} \frac{\partial w_h}{\partial t} \right) + \boldsymbol{\sigma}_h \cdot \left(\nabla w_h + \frac{\partial \boldsymbol{\tau}_h}{\partial t} \right) \right] dV \\
 & + \int_{\partial K} \left[(\hat{v}_h \boldsymbol{\tau}_h + \hat{\boldsymbol{\sigma}}_h w_h) \cdot \mathbf{n}_K^x + (\hat{\boldsymbol{\sigma}}_h \cdot \boldsymbol{\tau}_h + c^{-2} \hat{v}_h w_h) n_K^t \right] dS = \int_K f w_h dV
 \end{aligned}$$

where $(\mathbf{n}_K^x, n_K^t) \in \mathbb{R}^{d+1}$ denotes the unit normal vector to ∂K pointing outside K

global DG formulation

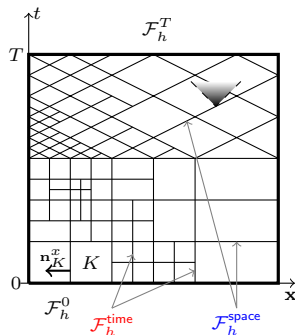
$$\text{add over all } K \in \mathcal{T}_h \rightarrow \mathcal{A}_{\text{DG}}(v_h, \boldsymbol{\sigma}_h; w_h, \boldsymbol{\tau}_h) = \ell_{\text{DG}}(w_h, \boldsymbol{\tau}_h)$$

assumption on \mathcal{T}_h each internal face F is either

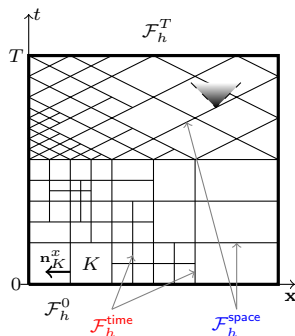
- space-like: $c |\mathbf{n}_F^x| < n_F^t$ ($F \subset \mathcal{F}_h^{\text{space}}$), or
- time-like: $n_F^t = 0$ ($F \subset \mathcal{F}_h^{\text{time}}$)

$$\mathcal{F}_h^0 := \Omega \times \{0\}, \quad \mathcal{F}_h^T := \Omega \times \{T\}$$

$$\mathcal{F}_h^\partial := \partial\Omega \times (0, T)$$



- across **space-like** faces, the information is transported from t^- to t^+
- across **time-like** faces, there is no directionality



assumptions on the numerical fluxes

$$\widehat{v}_h := \begin{cases} v_h^- & \\ \{\{v_h\}\} + \beta [\sigma_h]_{\mathbf{N}} & \\ v_0 & \\ 0 & \end{cases} \quad \widehat{\sigma}_h := \begin{cases} \sigma_h^- & \text{on } \mathcal{F}_h^{\text{space}} \cup \mathcal{F}_h^T & \text{(upwind fluxes)} \\ \{\{\sigma_h\}\} + \alpha [v_h]_{\mathbf{N}} & \text{on } \mathcal{F}_h^{\text{time}} & \text{(DG-elliptic fluxes)} \\ \sigma_0 & \text{on } \mathcal{F}_h^0 \\ \sigma_h - \alpha v \mathbf{n}_{\Omega}^x & \text{on } \mathcal{F}_h^{\partial} \end{cases}$$

$$\alpha, \beta \in L^\infty(\mathcal{F}_h^{\text{time}} \cup \mathcal{F}_h^{\partial}); \quad \alpha = \beta = 0 \quad [\text{Egger \& al., 2014}], \quad \alpha\beta \geq \frac{1}{4} \quad [\text{Monk, Richter, 2005}]$$

recall the definition of the wave operator $\mathcal{L}_{\text{wave}}(w, \boldsymbol{\tau}) := \left(\nabla \cdot \boldsymbol{\tau} + c^{-2} \frac{\partial w}{\partial t}, \nabla w + \frac{\partial \boldsymbol{\tau}}{\partial t} \right)$

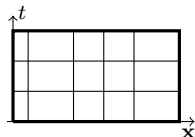
assumption on $\mathbf{V}_p(\mathcal{T}_h)$

for all $(w_h, \boldsymbol{\tau}_h) \in \mathbf{V}_p(\mathcal{T}_h)$, $\mathcal{L}_{\text{wave}}(w_h, \boldsymbol{\tau}_h) \in \mathbf{V}_p(\mathcal{T}_h)$

this is satisfied, e.g., if the restriction of $\mathbf{V}_p(\mathcal{T}_h)$ to each mesh element is made of

- total degree space-time polynomials $\mathbb{P}_{\mathbf{x},t}^P$,
- tensor product (in time) polynomials $\mathbb{P}_{\mathbf{x}}^P \times \mathbb{P}_t^P$,
- Trefftz polynomials $\mathcal{L}_{\text{wave}}(w_h, \boldsymbol{\tau}_h) = (0, \mathbf{0})$

- case of tensor product (in time) meshes*



key property: coercivity in seminorm

$$\mathcal{A}_{\text{DG}}(v_h, \boldsymbol{\sigma}_h; v_h, \boldsymbol{\sigma}_h) = |(v_h, \boldsymbol{\sigma}_h)|_{\text{DG}}^2$$

DG seminorm

$$\begin{aligned} |(w, \boldsymbol{\tau})|_{\text{DG}}^2 &= \frac{1}{2} \left\| c^{-1} \llbracket w \rrbracket_t \right\|_{L^2(\mathcal{F}_h^{\text{space}})}^2 + \frac{1}{2} \left\| \llbracket \boldsymbol{\tau} \rrbracket_t \right\|_{L^2(\mathcal{F}_h^{\text{space}})_d}^2 + \left\| \alpha^{\frac{1}{2}} \llbracket w \rrbracket_{\text{N}} \right\|_{L^2(\mathcal{F}_h^{\text{time}})_d}^2 + \left\| \beta^{\frac{1}{2}} \llbracket \boldsymbol{\tau} \rrbracket_{\text{N}} \right\|_{L^2(\mathcal{F}_h^{\text{time}})}^2 \\ &+ \frac{1}{2} \left\| c^{-1} w \right\|_{L^2(\mathcal{F}_h^0 \cup \mathcal{F}_h^T)}^2 + \frac{1}{2} \left\| \boldsymbol{\tau} \right\|_{L^2(\mathcal{F}_h^0 \cup \mathcal{F}_h^T)_d}^2 + \left\| \alpha^{\frac{1}{2}} w \right\|_{L^2(\mathcal{F}_h^\partial)}^2 \end{aligned}$$

by adapting [Monk, Richter, 2005], one deduces well-posedness, with **no condition on h_t**

*The case of general, admissible meshes requires minor, technical changes.

Proof of well-posedness

$$(*) \quad \mathcal{A}_{\text{DG}}(v_h, \sigma_h; w_h, \tau_h) = 0 \quad \forall (w_h, \tau_h) \in \mathbf{V}_p(\mathcal{T}_h) \quad \Rightarrow (v_h, \sigma_h) = (0, \mathbf{0})$$

i) take $(w_h, \tau_h) = (v_h, \sigma_h) \rightarrow 0 = \mathcal{A}_{\text{DG}}(v_h, \sigma_h; v_h, \sigma_h) = |(v_h, \sigma_h)|_{\text{DG}}^2$

\Rightarrow all jumps and boundary traces of v_h and σ_h are zero

ii) using *i)* in $(*)$ and integrating by parts give

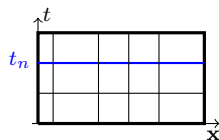
$$\int_K \left[\underbrace{\left(\nabla \cdot \sigma_h + c^{-2} \frac{\partial v_h}{\partial t} \right)}_{\text{}} w_h + \underbrace{\left(\nabla v_h + \frac{\partial \sigma_h}{\partial t} \right)}_{\text{}} \cdot \tau_h \right] dV = 0 \quad \forall (w_h, \tau_h) \in \mathbf{V}_p(\mathcal{T}_h)$$

iii) take $w_h = \uparrow$ and $\tau_h = \uparrow$

$\Rightarrow (v_h, \sigma_h)$ solves the homogeneous wave problem in each K

iv) steps *i)* and *iii)* imply that $(v_h, \sigma_h) = (0, \mathbf{0})$

- case of tensor product (in time) meshes



error estimates (with no condition on h_t)

assume that all the traces of the analytical solution on mesh faces are in L^2
 → error bound in the L^2 norm in space at every discrete time t_n :

$$\begin{aligned} \|c^{-1}(v - v_h)\|_{L^2(\Omega \times \{t_n\})} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L^2(\Omega \times \{t_n\})^d} &\leq |(v, \boldsymbol{\sigma}) - (v_h, \boldsymbol{\sigma}_h)|_{\text{DG}(Q_n)} \\ &\lesssim |(v, \boldsymbol{\sigma}) - \underbrace{\Pi(v, \boldsymbol{\sigma})}_{\in \mathbf{V}_p(\mathcal{T}_h)}|_{\text{DG}^+} \end{aligned}$$

(proven by restricting to partial space-time cylinders $Q_n = \Omega \times (0, t_n)$)

projector Π

- total degree space-time polynomials $\mathbb{P}_{\mathbf{x},t}^p$: construction in [Monk, Richter, 2005]
- tensor product (in time) polynomials $\mathbb{P}_{\mathbf{x}}^p \times \mathbb{P}_t^p$: L^2 projection [3]
- Trefftz polynomials: best approximation [1]

$$\text{recall: } \mathcal{L}_{\text{wave}}(w, \boldsymbol{\tau}) = \left(\nabla \cdot \boldsymbol{\tau} + c^{-2} \frac{\partial w}{\partial t}, \nabla w + \frac{\partial \boldsymbol{\tau}}{\partial t} \right)$$

Treffitz spaces

continuous spaces	$\mathbf{T}(K) := \{(w, \boldsymbol{\tau}) \in H^1(K) : \mathcal{L}_{\text{wave}}(w, \boldsymbol{\tau}) = (0, \mathbf{0})\}$ $\mathbf{T}(\mathcal{T}_h) := \{(w, \boldsymbol{\tau}) \in H^1(\mathcal{T}_h)^{1+d} : (w, \boldsymbol{\tau}) _K \in \mathbf{T}(K) \quad \forall K \in \mathcal{T}_h\}$
discrete spaces	$\mathbf{V}_p(K) \subset \mathbf{T}(K), \quad \mathbf{V}_p(\mathcal{T}_h) \subset \mathbf{T}(\mathcal{T}_h)$

in each element K , the *linear* operator $\mathcal{L}_{\text{wave}}$ is

- homogeneous (= all terms are derivatives of the *same* order)
- with constant coefficients

\Rightarrow Taylor polynomials of (smooth) functions in $\ker(\mathcal{L}_{\text{wave}})$ are in $\ker(\mathcal{L}_{\text{wave}})$

therefore, we can choose $\mathbf{V}_p(K) \subset \mathbf{T}(K)$

- as a subspace of the **polynomial** space $\mathbb{P}^p(K)^{1+d}$
- with the same order of **approximation** in h as $\mathbb{P}^p(K)^{1+d}$ for functions in $\ker(\mathcal{L}_{\text{wave}})$

Example:

$$\mathbb{T}^p(K) := \left\{ u \in \mathbb{P}^p(K) : -\Delta u + c^{-2} \frac{\partial^2 u}{\partial t^2} = 0 \right\} \quad \mathbf{V}_p(K) := \left(\frac{\partial \mathbb{T}^{p+1}(K)}{\partial t}, -\nabla(\mathbb{T}^{p+1}(K)) \right)$$

- reduction of number of degrees of freedom to that of a d -dimensional problem

	Treffitz polyn. $\mathbb{T}^p(K)$	full polyn. $\mathbb{P}^p(K)$
$d + 1 = 1 + 1$	$2p + 1$	$\frac{1}{2}(p + 1)(p + 2)$
$d + 1 = 2 + 1$	$(p + 1)^2$	$\frac{1}{6}(p + 1)(p + 2)(p + 3)$
$d + 1 = 3 + 1$	$\frac{1}{6}(p + 1)(p + 2)(2p + 3)$	$\frac{1}{24}(p + 1)(p + 2)(p + 3)(p + 4)$
	$\mathcal{O}(p^d)$	$\mathcal{O}(p^{d+1})$

$$\dim(\mathbb{T}^p(K)) = \mathcal{O}(p^d) \ll \dim(\mathbb{P}^p(K)) = \mathcal{O}(p^{d+1})$$

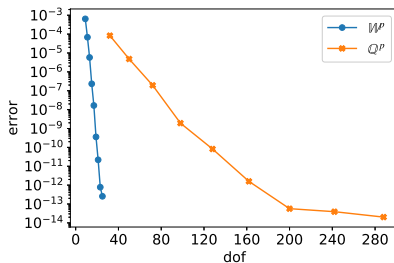
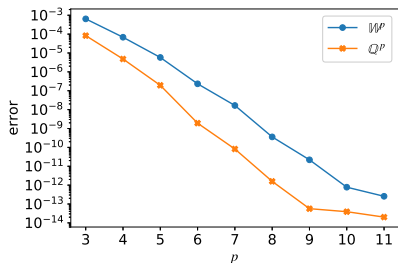
- same orders of approximation in h as with the full polynomial spaces



Netgen/NGSolve

[2] P., Schöberl, Stocker, Wintersteiger, 2020

$d = 1$, smooth solution, Cartesian mesh; Treffitz (blue) and Q^p (orange) polynomials



p -version: error (in $L^2(\Omega \times \{T\})$) vs. polynomial degree (left) and number of dof.s (right)

[1] Moiola, P., 2018

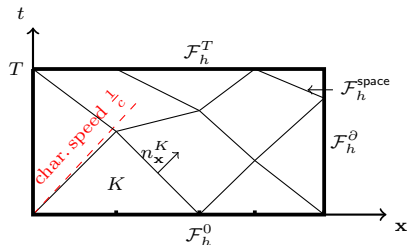
- in $\mathbf{T}(\mathcal{T}_h)$, the DG seminorm is actually a **norm**
- existence and uniqueness of solutions follow from $\mathcal{A}_{\text{DG}}(v_h, \boldsymbol{\sigma}_h; v_h, \boldsymbol{\sigma}_h) = |(v_h, \boldsymbol{\sigma}_h)|_{\text{DG}}^2$
- error bounds in the (spatial) L^2 norm on space-like interfaces (e.g. on $\Omega \times \{t_n\}$) and in DG norm also follow
- error bounds in a **global, mesh-independent norm** ($L^2(Q)$, in the best case scenario*) have also been proven in [1] by a modified duality argument from [Monk, Wang, 1999]

piecewise smooth coefficients: space-time **quasi-Trefftz** DG method

[Imbert-Gérard, Moiola, Stocker, 2020]

*i.e. for $d = 1$ or $d > 1$ and no time-like faces (for impedance b.c.) ;
in $H^{-1}(0, T; L^2(\Omega)) \times L^2(0, T; H^{-1}(\Omega)^d)$ for tensor product elements (with Dirichlet b.c.)

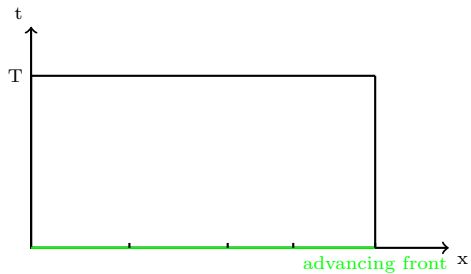
PDE-driven, front-advancing mesh construction technique

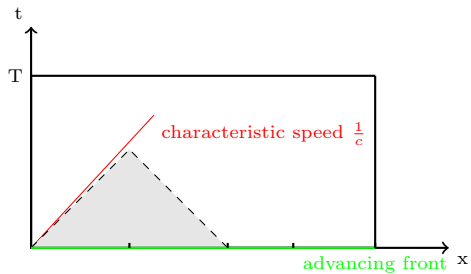


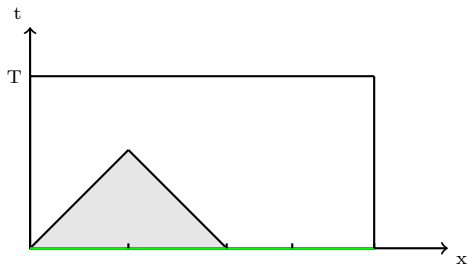
- progressively advancing in time and stacking tent-pitched objects on top of each other
- each tent is union of $(d + 1)$ -dimensional simplices
- the high of each tent (local advancement in time) is chosen so that the causality constraint of the PDE is respected (*local CFL condition*)

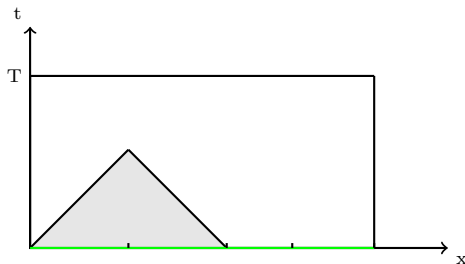
→ the PDE is explicitly solvable within each tent

[Falk, Richter, 1999], [Yin, Acharya, Sobh, Haber, Tortorelli, 2000] [Üngör, Sheffer, 2002],
 [Monk, Richter, 2005], [Abedi, Petracovici, Haber, 2006], ...,
 [Gopalakrishnan, Monk, Sepúlveda, 2015], [Gopalakrishnan, Schöberl, Wintersteiger, 2017, 2019]









solution at the tent bottom \rightarrow solution at the tent top

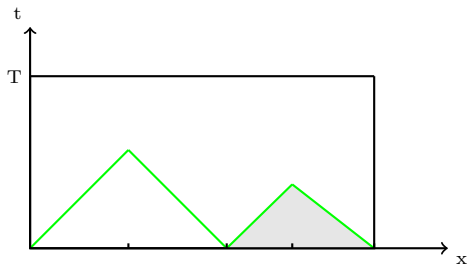
- Trefftz (no volume terms): solution of local problems [1], [2]; for an interior tent:

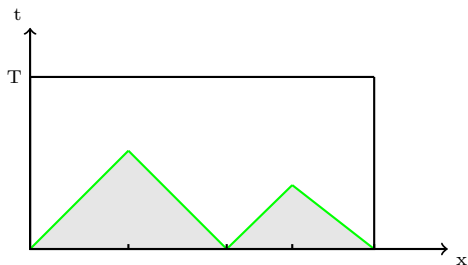
$$\begin{aligned} & \int_{\partial K^{\text{top}}} \left((v_h \boldsymbol{\tau}_h + \boldsymbol{\sigma}_h w_h) \cdot \mathbf{n}_K^{\mathbf{x}} + (\boldsymbol{\sigma}_h \cdot \boldsymbol{\tau}_h + c^{-2} v_h w_h) n_K^t \right) dS \\ &= - \int_{\partial K^{\text{bot}}} \left((v_h^{\text{bot}} \boldsymbol{\tau}_h + \boldsymbol{\sigma}_h^{\text{bot}} w_h) \cdot \mathbf{n}_K^{\mathbf{x}} + (\boldsymbol{\sigma}_h^{\text{bot}} \cdot \boldsymbol{\tau}_h + c^{-2} v_h^{\text{bot}} w_h) n_K^t \right) dS \end{aligned}$$

- mapping + RK or Taylor

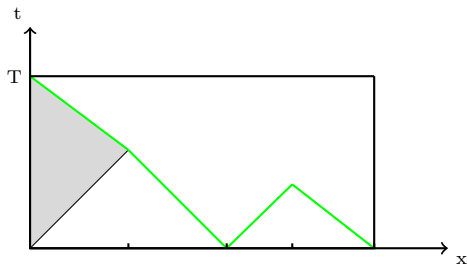
[Gopalakrishnan, Schöberl, Wintersteiger, 2017, 2019]

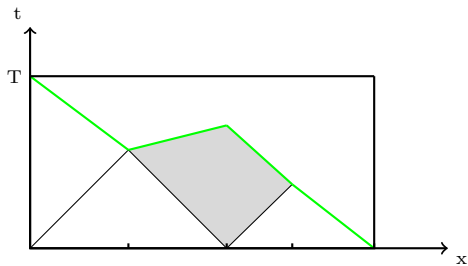


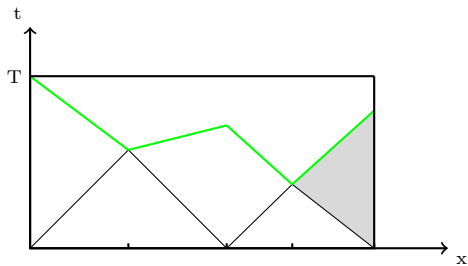


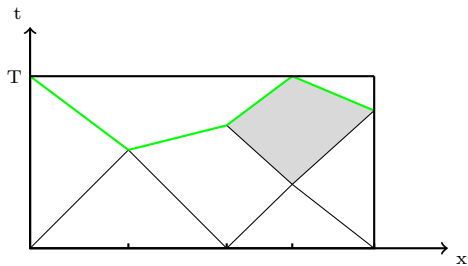


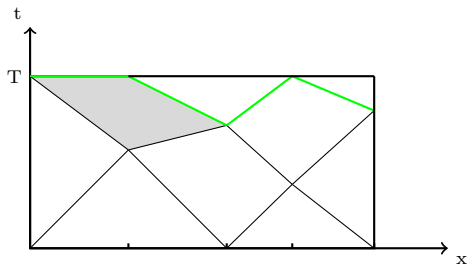
the solution within these two tents can be evolved in parallel

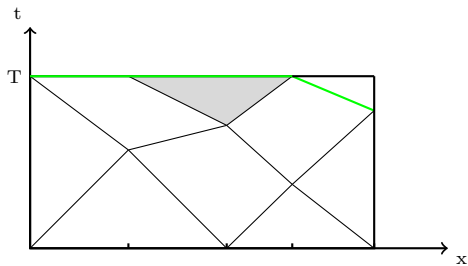


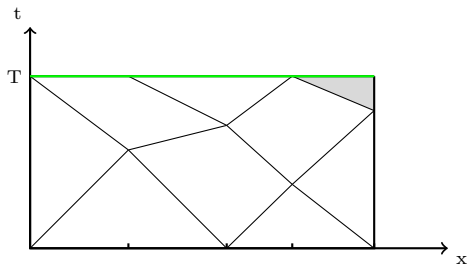




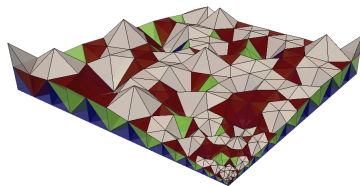
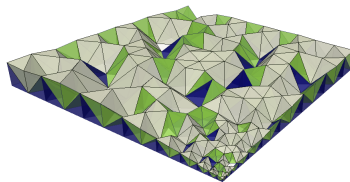
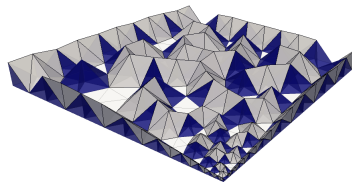
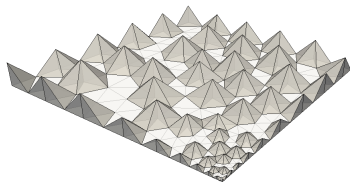








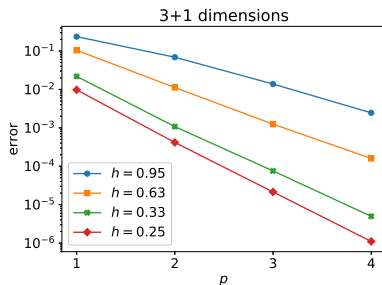
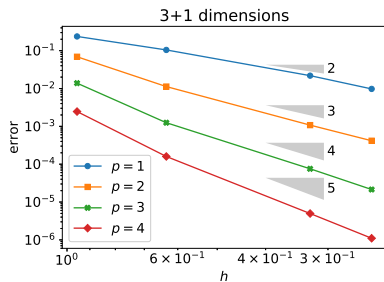
$d = 2$ and refined mesh towards a corner



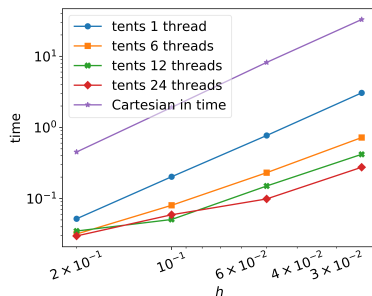
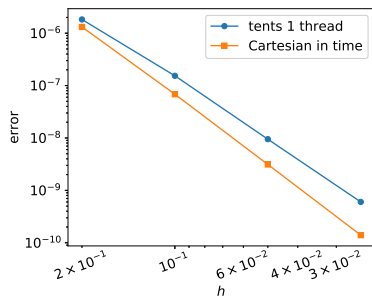


Netgen/NGSolve

[2] P., Schöberl, Stocker, Wintersteiger, 2020

 $d = 3$, smooth solution, Trefftz on tent-pitched meshes; h - and p -versionconvergence of order $p + 1$ in h (left) and exponential convergence in p (right)

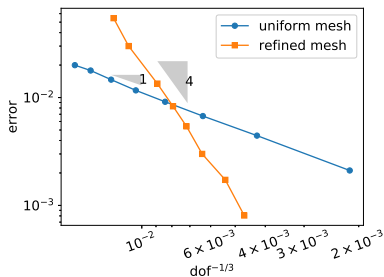
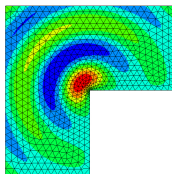
$d = 2$, smooth solution, Trefftz; tensor product (in time) meshes and tent-pitching



$d = 2$, singular solution, Trefftz on tent-pitched meshes

$$U(r, \varphi, t) = \cos(10t) \sin(\nu \varphi) J_\nu(10r)$$

$$\nu = \frac{2}{3} \rightarrow U \in H^{\frac{5}{3}-\varepsilon}(Q)$$



$p = 3$; spatial mesh at $t = 0$: uniform (blue) or with corner refinement (orange)

Regularity theory in 2D

[Kokotov, Plamenevskii, 1999, 2004], [Luong, Tung, 2015]

$\mathcal{S} := \{\mathbf{c}_j, j = 1, \dots, M\}$ set of all vertices of $\{\Omega_i\}$, in which c is piecewise constant

acoustic waves exhibit **conical singularities at \mathcal{S}** : regularity results are given in weighted Sobolev spaces in Ω with weight function

$$\Phi_{\delta}(\mathbf{x}) = \prod_{j=1}^M |\mathbf{x} - \mathbf{c}_j|^{\delta_j}, \quad \delta_j \in [0, 1]^*$$

e.g. $|u|_{H_{\delta}^{1,1}(\Omega)} := \|\Phi_{\delta} \nabla u\|_{L^2(\Omega)^2} \quad (H_{\delta}^{1,1}(\Omega) \not\subset H^1(\Omega))$

(used for the analysis of DG + time-stepping [Müller, Schötzau, Schwab, 2018]).

Example: if $v_0, u_0 \in C_0^{\infty}(\Omega)$, $\sigma_0 = -\nabla u_0$, $f \in C_0^{\infty}(Q)$, $\exists \delta \in [0, 1]^M$ such that $\forall k_t, k_x \in \mathbb{N}$,

$$v \in C^{k_t-1}([0, T]; H_{\delta}^{k_x+1,2}(\Omega)) \quad \sigma \in C^{k_t}([0, T]; H_{\delta}^{k_x,1}(\Omega))^2$$

[Müller, 2017]

$$* \|u\|_{H_{\delta}^{k,\ell}(\Omega)}^2 := \|u\|_{H^{\ell-1}(\Omega)}^2 + |u|_{H_{\delta}^{k,\ell}(\Omega)}^2, \quad |u|_{H_{\delta}^{k,\ell}(\Omega)}^2 := \sum_{m=\ell}^k \int_{\Omega} \left(\Phi_{\delta+m-\ell} \sum_{|\alpha|=m} |D^{\alpha} u|^2 \right) dx.$$

[3] Bansal, Moiola, P., Schwab, 2020

tensor product (in time) space-time meshes

- time mesh: $\mathcal{T}_{h_t}^t$ partition of $(0, T)$ into N intervals I_n
- spatial meshes: for each $1 \leq n \leq N$, $\mathcal{T}_{h_{\mathbf{x}}, n}^{\mathbf{x}}$ shape-regular mesh of Ω
 - with non-degenerating faces
 - aligned with $\{\Omega_i\}$
 - each mesh element touches at most one element of \mathcal{S}
- space-time mesh: $\mathcal{T}_h := \mathcal{T}_h(Q) := \{K = K_{\mathbf{x}} \times I_n : K_{\mathbf{x}} \in \mathcal{T}_{h_{\mathbf{x}}, n}^{\mathbf{x}}, 1 \leq n \leq N\}$

abstract error analysis (Galerkin error $\lesssim L^2$ projection error)

the critical solution regularity is the regularity in space of σ close to any $\mathbf{c} \in \mathcal{S}$:

if F is a time-like face of an element K adjacent to a corner \mathbf{c} ,
then $\sigma|_F \in L^1(F)^2$, not necessarily $L^2(F)^2$

→ modify the DG seminorm and apply Hölder in L^1 - L^∞ (instead of Cauchy-Schwarz)

[Wihler, 2002]

mesh grading in space (like in the elliptic case)

lack of smoothness \rightarrow loss in the accuracy of the L^2 projection of the solution in the elements $K = K_{\mathbf{x}} \times I_n$ that are close to any $\mathbf{c} \in \mathcal{S}$

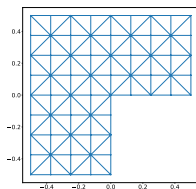
a reduction of the **size of $K_{\mathbf{x}}$** depending on

- the corner weight $\delta_{\mathbf{c}}$
- the polynomial approximation degree p_K

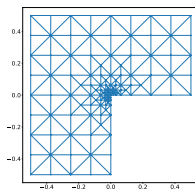
can restore the largest possible convergence rates

suitable graded spatial meshes $\mathcal{T}_{h_{\mathbf{x}}}^{\mathbf{x}}$ can be constructed from a quasi-uniform initial mesh $\mathcal{T}_0^{\mathbf{x}}$ of Ω of size $h_{\mathbf{x}}$ by J levels of local bisection refinement ($J = J(h_{\mathbf{x}}, \delta_{\mathbf{c}}, p_K)$)

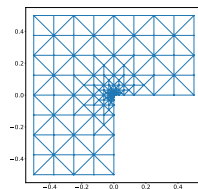
[Gaspaz, Morin, 2009]



uniform, $h_{\mathbf{x}} = 0.25$



graded, $p = 1$



graded, $p = 2$

assume, for simplicity, constant c , uniform p

- fix $h_t, h_{\mathbf{x}} > 0$, and construct the uniform mesh $\mathcal{T}_{h_t}^t$ and the locally refined mesh $\mathcal{T}_{h_{\mathbf{x}}}^{\mathbf{x}}$
- on any time-like face F , define the numerical flux parameters as $\alpha = \beta^{-1} = \frac{h_{\mathbf{x}}}{c |F_{\mathbf{x}}|}$
- assume that $ch_t \simeq h_{\mathbf{x}}$ ($h_{\mathbf{x}}$ is the size of *the largest* element of $\mathcal{T}_{h_{\mathbf{x}}}^{\mathbf{x}}$)

error bounds

for every discrete time t_n , we have

$$\|c^{-1}(v - v_h)\|_{L^2(\Omega \times \{t_n\})} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L^2(\Omega \times \{t_n\})^2} \leq |(v, \boldsymbol{\sigma}) - (v_h, \boldsymbol{\sigma}_h)|_{\text{DG}(Q_n)} \lesssim h^{p+\frac{1}{2}}$$

(same convergence rates as for smooth solutions)

Remark: $\dim(\mathbf{V}(\mathcal{T}_h)) = \mathcal{O}(h^{-3})$ (like for a $(2+1)$ -dimensional elliptic problem)

Q: Can we obtain the same convergence rates with $\mathcal{O}(h^{-2})$ degrees of freedom?
(like for a 2-dimensional elliptic problem)

- the assumption $cht \simeq h_x$ is necessary to obtain the highest convergence rates
- stability of the DG formulation and best approximation-type estimates are valid with **no condition on h_t**
- the solutions obtained with anisotropic (in time) space-time meshes are not accurate, still they contain meaningful information

$\mathcal{T}_{(0,0)}$ coarsest space-time mesh

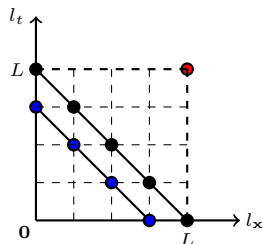
$\mathcal{T}_{(L,L)}$ finest space-time mesh (red)

$\mathcal{T}_{(l_x, l_t)}$ intermediate meshes

$\mathbf{w}_{(l_x, l_t)} := (v_{(l_x, l_t)}, \boldsymbol{\sigma}_{(l_x, l_t)})$ solution on $\mathcal{T}_{(l_x, l_t)}$

$\mathbf{w}_F := \mathbf{w}_{(L,L)}$ full space-time solution

$\mathbf{w}_S := \sum_{l=0}^L \mathbf{w}_{(l, L-l)} - \sum_{l=1}^L \mathbf{w}_{(l-1, L-l)}$ “sparse” solution



[Bungartz, Griebel, 2004]

Count of degrees of freedom (h -version): # d.o.f.s for $\mathbf{w}_F \lesssim 2^{3L} = \mathcal{O}(h_L^{-3})$

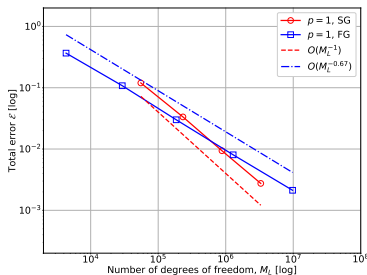
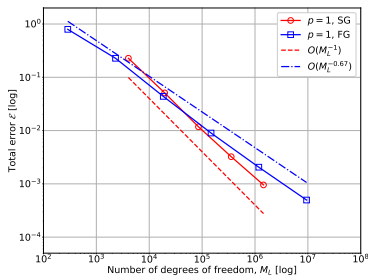
d.o.f.s for $\mathbf{w}_S \lesssim 2^{2L} = \mathcal{O}(h_L^{-2})$

(\simeq one time-step on the finest spatial mesh)



FEniCS

[3] Bansal, Moiola, P., Schwab, 2020

 Expected convergence rates: full $\mathcal{O}(\text{Ndofs})^{-\frac{p+1/2}{3}}$, sparse $\mathcal{O}(\text{Ndofs})^{-\frac{p+1/2}{2}}$


$p = 1$, full (blue), sparse (red)

smooth solution, uniform meshes (left), conical singularity, spatially graded meshes (right)

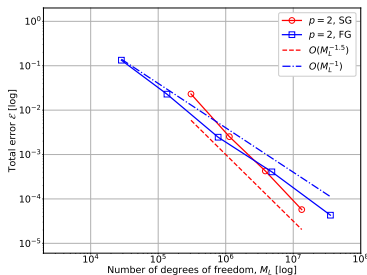
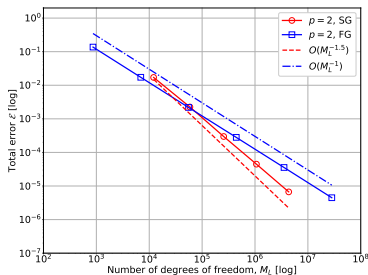
Obtained convergence rates: full $\mathcal{O}(\text{Ndofs})^{-\frac{p+1}{3}}$, sparse $\mathcal{O}(\text{Ndofs})^{-\frac{p+1}{2}}$



FEniCS

[3] Bansal, Moiola, P., Schwab, 2020

Expected convergence rates: full $\mathcal{O}(\text{Ndofs})^{-\frac{p+1/2}{3}}$, sparse $\mathcal{O}(\text{Ndofs})^{-\frac{p+1/2}{2}}$



$p = 2$, full (blue), sparse (red)

smooth solution, uniform meshes (left), conical singularity, spatially graded meshes (right)

Obtained convergence rates: full $\mathcal{O}(\text{Ndofs})^{-\frac{p+1}{3}}$, sparse $\mathcal{O}(\text{Ndofs})^{-\frac{p+1}{2}}$



Thank you for your attention!