The Geometry of Random Spherical Eigenfunctions

Domenico Marinucci
Dipartimento di Matematica
Università di Roma Tor Vergata
8th European Congress of Mathematics, 2021

Based on joint works with
Igor Wigman (King's College London)
Valentina Cammarota (Sapienza)
Maurizia Rossi (Milano Bicocca)

June 25, 2021

Spherical Eigenfunctions on $\mathbb{S}^2$

Spherical eigenfunctions on $\mathbb{S}^2$ (*spherical harmonics*) are defined by the *Helmholtz equation*

$$\Delta_{\mathbb{S}^2} f_\ell + \lambda_\ell f_\ell = 0 \ , \ f_\ell : \mathbb{S}^2 \to \mathbb{R} \ , \ \ell = 1, 2, ...,$$

where $\lambda_\ell = \ell(\ell + 1)$, $\ell = 1, 2, 3...$, and

$$\Delta_{\mathbb{S}^2} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left\{\sin\theta\frac{\partial}{\partial\theta}\right\} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} \ , \ 0 \leq \theta \leq \pi \ , \ 0 \leq \varphi < 2\pi \ .$$

Geometry of random eigenfunctions - Motivations

- Zero sets of eigenfunctions have been studied from the beginning of the XIX century (vibration modes of a membrane - Chladni (1808), S. Germain (1816), Kirchoff (1850) - see A.Logunov's talk on wednesday)
- Quantum mechanics - Berry's ansatz (1977,2002)
- Seminal papers on nodal sets of random spherical harmonics by Nazarov and Sodin (2009) and Wigman (2010)
- Fourier components of spherical random fields - strong motivations from Cosmology (CMB)

Gaussian Random eigenfunctions

Consider the isotropic Gaussian random field

$$f : \Omega \times \mathbb{S}^2 \to \mathbb{R},\ f(gx) \triangleq f(x),\ g \in SO(3);$$

then for some $\{\gamma_\ell\}_{\ell=0,1,2,\ldots} : \sum \gamma_\ell^2 < \infty$ (Spectral Representation Theorem)

$$f(x) = \sum_\ell \gamma_\ell f_\ell(x),\ \text{in } L^2(\Omega)$$

$$\mathbb{E} f_\ell(x) = 0,\ \mathbb{E} f_\ell(x) f_\ell(y) = P_\ell(\langle x, y\rangle)\ ,$$

and we have introduced Legendre Polynomials

$$P_\ell(t) := \frac{1}{2^\ell \ell!} \frac{d^\ell}{dt^\ell}(t^2 - 1)^\ell,$$

$$P_\ell'(1) = \frac{\ell \times (\ell + 1)}{2} = \frac{\lambda_\ell}{2}.$$

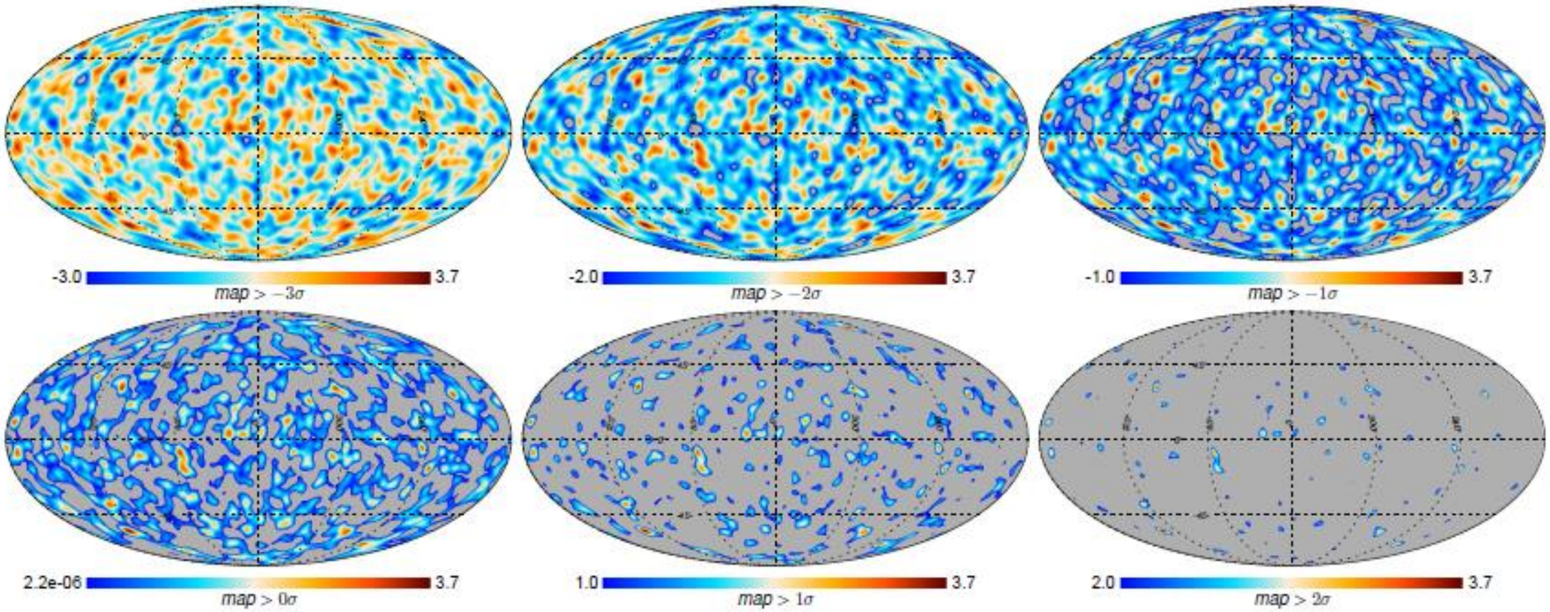
Excursion Sets

Let M be a general Riemannian manifold (in our case $M = \mathbb{S}^2$). Consider the *excursion sets*

$$A_u(f, M) = \{x \in M : f(x) \geq u\}$$

Our main goal in this talk is to study the geometry of these excursion sets in the case of Gaussian random eigenfunctions

Excursion sets



LKC in dimension 2 - Tubes

Lipschitz-Killing Curvatures of Excursion Sets

By the (Steiner's) Tube Formula:

$$\mu(Tube(M,\rho)) = \sum_{j=0}^{n=dim(M)} \omega_j \mathcal{L}_{n-j}(M)\rho^j$$

for

$$Tube(M,\rho) = \{t \in \mathbb{R}^N : dist(M,x) \leq \rho\}$$

where ω_j is the volume of a unit ball in $\mathbb{R}^j$, $\omega_j = \frac{\pi^{j/2}}{\Gamma(\frac{j}{2}+1)}$.

Lipschitz-Killing Curvatures II

LKCs depend on the Riemannian metric, and are a measure of the k-dimensional size of the Riemannian manifold M. In particular, in dimension 2

- $\mathcal{L}_0(A_u(f))$ is the Euler-Poincarè characteristic (minima+maxima-saddles) of the excursion regions, i.e. the third Minkowski functional (2 for the sphere).
- $\mathcal{L}_1(A_u(f))$ is half the boundary length of the excursion regions, e.g. the second Minkowski functional (0 for the sphere).
- $\mathcal{L}_2(A_u(f))$ is the area of the excursion regions, e.g. the first Minkowski functional (4π for the sphere).

Some Notation

Let us write (*Flag Coefficients*)

$$\begin{bmatrix} i+k \\ k \end{bmatrix} = \binom{i+k}{k} \frac{\omega_{i+k}}{\omega_k \omega_i},$$

$$\rho_k(u) = (2\pi)^{-k/2} H_{k-1}(u)\phi(u) = (2\pi)^{-k/2}(-1)^{k-1}\frac{d^k}{du^k}\Phi(u),$$

$$\phi(u) = \frac{d}{du}\Phi(u) = (2\pi)^{-1/2} exp(-u^2/2),$$

and H_j denotes the Hermite polynomials: $H_0(u) = 1$, $H_1(u) = u$, $H_2(u) = u^2 - 1$, $H_3(u) = u^3 - 3u$,....

Gaussian Kinematic Formula (GKF)

Due to Taylor and Adler (2007), it allows to evaluate expected values of Lipschitz-Killing curvatures for excursion regions under very general (but Gaussian) circumstances.

$$\mathbb{E}\mathcal{L}_i^f(A_u(f,M)) = \sum_{k=0}^{dimM-i} \begin{bmatrix} i+k \\ k \end{bmatrix} \mathcal{L}_{i+k}^f(M)\rho_k(u)$$

Note that $\rho_k(u)$ does not depend on f, M and $\mathcal{L}_{i+k}^f(M)$ does not depend on u.

Gaussian Kinematic Formula (GKF)

The Lipschitz-Killing Curvatures $\mathcal{L}_i^f$ are computed under the metric induced by the covariance structure of the field; for every $X_p, Y_p \in \mathbb{T}_p(M)$

$$g_f(X_p, Y_p) := \mathbb{E}[X_p f \times Y_p f].$$

For isotropic fields, this is the Euclidean metric scaled by the square root of the covariance function at the origin (for random spherical harmonics $\sqrt{\lambda_\ell/2}$)

$$g_{f_\ell}(X_p, Y_p) = \sqrt{\lambda_\ell/2}\langle X_p, Y_p\rangle_{\mathbb{R}^3}.$$

LKCs for Gaussian spherical harmonics

- First LKC (e.g. Euler-Poincarè characteristic)

$$\mathbb{E}\mathcal{L}_0(A_u(f(x), S^2)) = 2\{1 - \Phi(u)\} + 4\pi \times (\lambda_\ell/2)\frac{1}{2\pi}u\phi(u) \ ;$$

- Second LKC (e.g., half the boundary length)

$$\mathbb{E}\mathcal{L}_1(A_u(f(x), S^2)) = 4\pi \times \sqrt{\lambda_\ell/2}\frac{\sqrt{\pi}}{2\sqrt{2}}\phi(u) \ ;$$

- Third LKC (e.g., area)

$$\mathbb{E}\mathcal{L}_2(A_u(f(x), S^2)) = 4\pi \times \{1 - \Phi(u)\} \ .$$

Our goal is to study the fluctuations (Variances and CLT, as $\ell \to \infty$) around these expected values.

Nodal Lines

Note that for $u = 0$ the *nodal length* has expected length

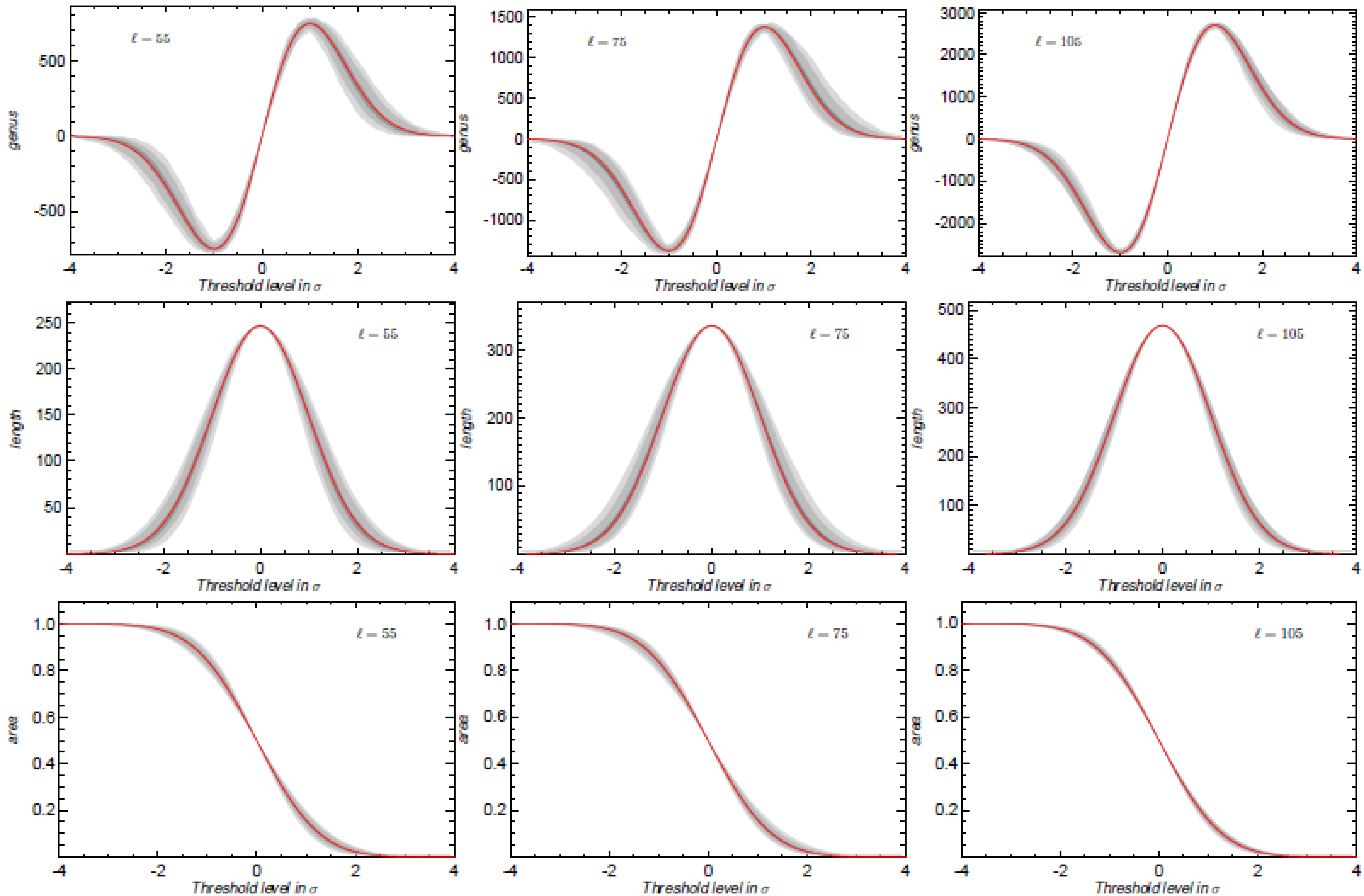
$$\mathbb{E}\{2\mathcal{L}_1(A_u(f_\ell(x), S^2))\} = \sqrt{2}\pi \times \sqrt{\lambda_\ell} \ ;$$

this is consistent with *Yau's Conjecture*:

$$c_1\sqrt{E} \leq \mathcal{H}(f_E^{-1}(0)) \leq c_2\sqrt{E} \ ,$$

see Logunov's talk on the zeroes of Laplace eigenfunctions (more to follow below).

Multipole space - Gaussian case



Analytical (blue curve) vs Simulation (black and grey. Grey Shades are 68, 95 and 99% percentiles estimated from 100 simulations.

2nd Order GKF

Theorem

(Cammarota and M (2018) - see also Rossi (2016), M and Wigman (2014)) As $\ell \to \infty$

$$\mathcal{L}_j(A_u(\mathbb{S}^2); f_\ell) - \mathbb{E}\left[\mathcal{L}_j(A_u(\mathbb{S}^2); f_\ell)\right]$$

$$= -\frac{1}{2}\begin{bmatrix} 2 \\ 2-j \end{bmatrix} u\rho'_{2-j}(u)\,(\lambda_\ell/2)^{(2-j)/2}\int_{\mathbb{S}^2} H_2(f_\ell(x))dx + R_{\ell;j}(.)\ ,$$

and we have also the following Variance asymptotics

$$Var\left\{\mathcal{L}_j(A_u(\mathbb{S}^2); f_\ell)\right\}$$

$$= \frac{1}{4}\left\{\begin{bmatrix} 2 \\ 2-j \end{bmatrix} u\rho'_j(u)\,(\lambda_\ell/2)^{(2-j)/2}\right\}^2 \times \frac{32\pi^2}{2\ell+1} + o_{\ell\to\infty}(\lambda_\ell^{2-j-1})\ .$$

Some Remarks

Some features of the previous result are worth discussing:

- The leading terms can be written in terms of the norm of the eigenfunctions (no need for gradient and Hessian)
- The asymptotic behaviour of all the LKC's is proportional to $\int_{\mathbb{S}^2} H_2(f_\ell(x))dx$. Full correlation in the high-energy limit $\ell \to \infty$;
- Full correlation across different levels u_1, u_2;
- The leading terms all disappear in the "nodal" case $u = 0$ - Berry's cancellation phenomenon (Wigman 2010).

Background: Wiener-Chaos Expansions

Consider (see Nourdin-Peccati 2011)

$$Y = G(Z),\ Z = N(0,1),\ ,\ \mathbb{E}\left[G(Z)^2\right] < \infty;$$

it is well-known that the following expansion holds, in the $L^2(\Omega)$ sense:

$$G(Z) = \sum_{q=0}^{\infty} \frac{J_q(G)}{q!} H_q(Z)\ , \tag{1}$$

where

$$J_q(G) := \mathbb{E}\left[G(Z) H_q(Z)\right]$$

.

Wiener Chaos

Diagram's (Wick) Formula

$$\mathbb{E}\left[H_{q_1}(Z_1)H_{q_2}(Z_2)\right] = \delta_{q_1}^{\delta_2} q_1! \left\{\mathbb{E}\left[Z_1 Z_2\right]\right\}^{q_1} .$$

Therefore

$$Var\left\{G(Z)\right\} = \sum_{q=0}^{\infty} \frac{J_q^2(G)}{q!} .$$

More generally, for $\{Z_1, ..., Z_j, ...\}$ i.i.d. standard Gaussian the q-th order Wiener chaos is

$$\mathcal{C}_q = span\{H_{q_1}(Z_1) \cdot ... \cdot H_{q_p}(Z_p)\} \text{ , } q_1 + ...q_p = q \text{ ;}$$

We have the orthogonal decomposition

$$L^2(\Omega) = \bigoplus_{q=0}^{\infty} \mathcal{C}_q .$$

Excursion Area

(M and Wigman 2011-2014) We can write

$$\mathcal{L}_2(A_u) = \int_{\mathbb{S}^2} \mathbb{I}_{[u,\infty)}(f_\ell(x))dx \ ,$$

$\mathbb{I}_{[u,\infty)}(.)$ denoting the indicator function. Also

$$\begin{aligned} J_q(\mathbb{I}_{[u,\infty)}(.)) &= \mathbb{E}\left[\mathbb{I}_{[u,\infty)}(Z)H_q(Z)\right] \\ &= \int_u^\infty H_q(z)\phi(z)dz = (-1)^q H_{q-1}(z)\phi(z) \ , \end{aligned}$$

where

$$(-1)H_{-1}(z)\phi(z) := 1 - \Phi(u) \ .$$

Excursion Area

Thus

$$\begin{aligned}\mathcal{L}_2(A_u) &= \int_{\mathbb{S}^2} \sum_{q=0}^{\infty} (-1)^q H_{q-1}(u)\phi(u) \frac{H_q(f_\ell(x))}{q!} dx \\ &= \sum_{q=0}^{\infty} \frac{(-1)^q}{q!} H_{q-1}(u)\phi(u) h_{\ell;q} \ , \ h_{\ell;q} = \int_{\mathbb{S}^2} H_q(f_\ell(x))dx \ ;\end{aligned}$$

also

$$Var\{\mathcal{L}_2(A_u)\} = \sum_{q=0}^{\infty} \frac{1}{(q!)^2} H_{q-1}^2(u)\phi^2(u) Var\{h_{\ell;q}\} \ .$$

Polyspectra

More explicitly, we have that

$$h_{\ell;0} = \int_{\mathbb{S}^2} H_0(f_\ell(x))dx = 4\pi \ ;$$

$$h_{\ell;1} = \int_{\mathbb{S}^2} f_\ell(x)dx = 0 \ ;$$

$$h_{\ell;2} = \int_{\mathbb{S}^2} (f_\ell^2(x) - 1)dx = \|f_\ell\|^2_{L^2(\mathbb{S}^2)} - \mathbb{E}\|f_\ell\|^2_{L^2(\mathbb{S}^2)} \ ;$$

$$h_{\ell;3} = \int_{\mathbb{S}^2} (f_\ell^3(x) - 3f_\ell(x))dx = \int_{\mathbb{S}^2} f_\ell^3(x)dx \ ;$$

$$h_{\ell;4} = \int_{\mathbb{S}^2} (f_\ell^4(x) - 6f_\ell^2(x) + 3)dx \ ,$$

$$\ldots$$

Variance

By Diagram Formula

$$\begin{aligned} Var\left\{h_{\ell;q}\right\} &= \int_{\mathbb{S}^2\times\mathbb{S}^2} \mathbb{E}\left\{H_q(f_\ell(x))H_q(f_\ell(y))\right\} dxdy \\ &= 8\pi^2 q! \int_0^\pi \left\{P_\ell(\cos\theta)\right\}^q \sin\theta d\theta \ ; \end{aligned}$$

The following asymptotics is crucial:

$$\begin{aligned} Var\left\{h_{\ell;q}\right\} &\sim \frac{Const}{\ell^2} \times \int_0^{\ell\pi} \frac{1}{\psi^{q/2}} \psi d\psi \\ &\sim \left\{ \begin{array}{c} O(\ell^{-1}) \text{ for } q=2 \\ O(\ell^{-2}\log\ell) \text{ for } q=4 \\ O(\ell^{-2}) \text{ for } q=3,5,... \end{array} \right. . \end{aligned}$$

Note that $h_{\ell;1} \equiv 0$ for all $\ell = 1, 2, ...,$

Reduction Principle

Dominant terms correspond to $q = 2$ when $H_1(u) \neq 0$, i.e., for $u \neq 0$; for $u = 0$ a phase transition occurs. Indeed

$$\mathcal{L}_2(A_u) - \mathbb{E}\left[\mathcal{L}_2(A_u)\right] = \frac{1}{2} H_1(u)\phi(u) h_{\ell;2} + O_p(\sqrt{\log \ell / \ell^2}) \ ,$$

and for $u \neq 0$

$$Var\left\{\mathcal{L}_2(A_u)\right\} \sim \left\{\frac{1}{2} H_1(u)\phi(u)\right\}^2 Var\left\{h_{\ell;2}\right\} \ , \text{ as } \ell \to \infty \ .$$

Note that

$$h_{\ell;2} = \int_{\mathbb{S}^2} \left\{f_\ell^2(x) - 1\right\} dx = \|f_\ell\|^2_{L^2(\mathbb{S}^2)} - \mathbb{E}\left[\|f_\ell\|^2_{L^2(\mathbb{S}^2)}\right] \ ,$$

e.g., fluctuations for $u \neq 0$ dominated by random norm of the eigenfunctions.

Other Lipschitz-Killing Curvatures

Likewise we have the expansion

$$2\mathcal{L}_1(A_u(\mathbb{S}^2); f_\ell) = \lim_{\varepsilon \to 0} \int_{\mathbb{S}^2} \|\nabla f_\ell(x)\| \, \delta_\varepsilon(f_\ell(x) - u) dx$$

ω-almost surely and in $L^2(\Omega)$, and similarly for the Euler-Poincaré Characteristic we have

$$\mathcal{L}_0(A_u(\mathbb{S}^2); f_\ell) = \lim_{\varepsilon \to 0} \int_{\mathbb{S}^2} \det \left\{\nabla^2 f_\ell(x)\right\} \delta_\varepsilon(\nabla f_\ell(x)) \mathbb{I}_{[u,\infty)}(f_\ell(x)) dx \ .$$

These expressions can be expanded into Hermite polynomials in $\left\{\nabla^2 f_\ell(.), \nabla f_\ell(.), f_\ell(.)\right\}$; the second-chaos however is a function of f_ℓ alone. Is this phenomenon more general?

Critical Values

Critical values are defined by

$$\mathcal{N}_u(f_\ell;\mathbb{S}^2)=\#\left\{x\in\mathbb{S}^2:\nabla f_\ell(x)=0 \text{ and } f_\ell(x)\geq u\right\} \ .$$

We have (Cammarota, M and Wigman 2016)

Theorem

$$\begin{aligned}\mathbb{E}\left[\mathcal{N}_u(f_\ell;\mathbb{S}^2)\right] &= \lambda_\ell g_1(u) \ , \\ g_1(u) &= \frac{1}{\sqrt{2\pi}}\int_u^\infty (2e^{-t^2}+(t^2-1)e^{-t^2/2})dt \\ &= u\phi(u)+\sqrt{2}(1-\Phi(\sqrt{2}u)) \ ,\end{aligned}$$

$$\begin{aligned}Var\left[\mathcal{N}_u(f_\ell;\mathbb{S}^2)\right] &= \frac{1}{4}\lambda_\ell^2 g_2^2(u) Var\left\{\int_{\mathbb{S}^2} H_2(f_\ell(x))dx\right\}+o_{\ell\to\infty}(\ell^3) \\ &= \frac{1}{4}\lambda_\ell^2 g_2^2(u)\frac{2(4\pi)^2}{2\ell+1}+o_{\ell\to\infty}(\ell^3) \ .\end{aligned}$$

Reduction principle for critical values

In Cammarota and M (2020) it was shown that

Theorem

$$\mathcal{N}_u(f_\ell; \mathbb{S}^2) - \mathbb{E}\left[\mathcal{N}_u(f_\ell; \mathbb{S}^2)\right]$$

$$= \frac{1}{2}\lambda_\ell g_2(u) \int_{\mathbb{S}^2} H_2(f_\ell(x))dx + o_p(\sqrt{Var\left[\mathcal{N}_u(f_\ell; \mathbb{S}^2)\right]}) \ ,$$

$$g_2(u) = \int_u^\infty \frac{1}{\sqrt{8\pi}} e^{-3t^2/2}(2 - 6t^2 - e^{-t^2}(1 - 4t + t^4))dt \ .$$

Correlation

As a consequence, for all $u \neq 0, 1$

$$\begin{aligned} & Corr^2 \left\{ \mathcal{N}_u(f_\ell; \mathbb{S}^2), \mathcal{L}_j(A_u(\mathbb{S}^2); f_\ell) \right\} \\ = \; & \frac{Cov^2 \left\{ \mathcal{N}_u(f_\ell; \mathbb{S}^2), \mathcal{L}_j(A_u(\mathbb{S}^2); f_\ell) \right\}}{Var \left\{ \mathcal{N}_u(f_\ell; \mathbb{S}^2) \right\} Var \left\{ \mathcal{L}_j(A_u(\mathbb{S}^2); f_\ell) \right\}} \to 1 \text{ , as } \ell \to \infty \text{ .} \end{aligned}$$

We also have, for all $u_1, u_2 \neq 0, 1$

$$Corr^2 \left\{ \mathcal{N}_{u_1}(f_\ell; \mathbb{S}^2), \mathcal{N}_{u_2}(f_\ell; \mathbb{S}^2) \right\} \to 1 \text{ , as } \ell \to \infty \text{ ,}$$

(asymptotically full correlation at any two non-zero thresholds u_1, u_2.)

Quantitative Central Limit Theorems

Wasserstein distance between two random variables X and Y:

$$D_W(X,Y) := \sup_{h \in Lip(1)} |\mathbb{E}h(X) - \mathbb{E}h(Y)| \ ,$$

$Lip(1)$ Lipschitz functions of constant 1.
Fourth-Moment Theorem (Nourdin and Peccati 2009,2011):
Taking $Z \sim N(0,1)$ and $X_n \in \mathcal{C}_q$

$$d_W\left(\frac{X_n - \mathbb{E}X_n}{\sqrt{Var(X_n)}}, Z\right) \leq \sqrt{\frac{2q-2}{3\pi q}} \sqrt{\mathbb{E}\left[\left(\frac{X_n - \mathbb{E}X_n}{\sqrt{Var(X_n)}}\right)^4\right] - 3} \ . \quad (2)$$

Similar results hold for Kolmogorov and Total Variation distances.

Quantitative Central Limit Theorems

Recall that $\{h_{\ell;q}\}$ belong to the q-th order Wiener chaos; hence it holds that:
As $\ell \to \infty$

$$d_W\left(\frac{h_{\ell;q} - \mathbb{E}\left[h_{\ell;q}\right]}{\sqrt{Var(h_{\ell;q})}}, Z\right) = \begin{cases} O(\frac{1}{\sqrt{\ell}}) \text{ for } q = 2, 3 \\ O(\frac{1}{\log \ell}) \text{ for } q = 4 \\ O(\ell^{-1/4}) \text{ for } q = 5, 6, \ldots \end{cases} .$$

Quantitative Central Limit Theorem

In view of the Reduction Principle we can show that

Theorem

As $\ell \to \infty$, for $u \neq 0$ $(j = 1, 2)$ and for $u \neq (0, 1)$ (for $j = 0$) we have that

$$d_W\left(\frac{\mathcal{L}_j(A_u(\mathbb{S}^2); f_\ell) - \mathbb{E}\left[\mathcal{L}_j(A_u(\mathbb{S}^2); f_\ell)\right]}{\sqrt{Var(\mathcal{L}_j(A_u(\mathbb{S}^2); f_\ell))}}, Z\right) = O(\ell^{-1/2})$$

and likewise for $u \neq 0, \pm 1$

$$d_W\left(\frac{\mathcal{L}_j(A_u(\mathbb{S}^2); f_\ell) - \mathbb{E}\left[\mathcal{L}_j(A_u(\mathbb{S}^2); f_\ell)\right]}{\sqrt{Var(\mathcal{L}_j(A_u(\mathbb{S}^2); f_\ell))}}, Z\right) = O(\ell^{-1/2}) \ .$$

A Higher-Dimensional Conjecture

Define the set of singular points

$$P_j := \left\{u \in \mathbb{R} : u\rho_j'(u) = 0\right\};$$

for instance,

$$P_0, P_1 = \{0\},$$

$P_2 = \{0, 1\}, P_3 = \left\{0, \pm\sqrt{3}\right\}, ...$
Gaussian random eigenfunctions on $\mathbb{S}^d$, e.g.

$$\Delta_{\mathbb{S}^d} f_{\ell;d} = -\lambda_{\ell;d} f_{\ell;d} \ , \ \lambda_{\ell;d} := \ell(\ell + d - 1) \ ;$$

Higher-dimensional Spherical Harmonics

$$\mathbb{E}\left[f_{\ell;d}\right]=0\ ,\ \mathbb{E}\left[f_{\ell;d}^2\right]=1\ ,\ \mathbb{E}\left[f_{\ell;d}(x)f_{\ell;d}(y)\right]=G_{\ell;d}(\langle x,y\rangle)\ ,$$

$G_{\ell;d}(.)$ Gegenbauer polynomial of order d (with $G_{\ell;d}(1)=1$); it is convenient to recall that

$$G'_{\ell;d}(1)=\frac{\lambda_{\ell;d}}{d}\ .$$

Dimension of the corresponding eigenspaces is

$$n_{\ell;d}=\frac{2\ell+d-1}{\ell}\binom{\ell+d-2}{\ell-1}\sim\frac{2}{(d-1)!}\ell^{d-1},\ \text{as } \ell\to\infty\ .$$

Conjecture - 2nd order GKF

We have

$$Var\left[\int_{\mathbb{S}^d} H_2(f_{\ell;d}(x))dx\right] = \frac{2s_d^2}{n_{\ell;d}} = \frac{(d+1)^2\omega_{d+1}^2}{n_{\ell;d}}$$
$$\sim \frac{(d+1)^2\omega_{d+1}^2(d-1)!}{\ell^{d-1}} \text{ as } \ell \to \infty .$$

We then propose the following Conjecture:
As $\ell \to \infty$, for all $k = 0, 1, ..., d$ we have that

$$\mathcal{L}_k(A_u(\mathbb{S}^d); f_\ell) - \mathbb{E}\left[\mathcal{L}_k(A_u(\mathbb{S}^d); f_\ell)\right]$$

$$= -\frac{1}{2}\left[\begin{matrix} 2 \\ d-k \end{matrix}\right]\rho'_{d-k}(u)u\left(\frac{\lambda_{\ell;d}}{d}\right)^{(d-k)/2}\int_{\mathbb{S}^d} H_2(f_{\ell;d}(x))dx + o_p(\sqrt{\ell^{d-2k+1}})$$

Nodal Cases: the Excursion Area

At $u = 0$ the leading second-order chaos disappears, and we are left with

$$Var\left\{\mathcal{L}_2(A_0(\mathbb{S}^d), f_\ell)\right\} = \frac{1}{\ell^2}\sum_{q=1}^{\infty}\frac{c_{2q+1}}{2\pi q!}H_{2q}^2(0) + o(\ell^{-2}) \ ,$$

where

$$c_{2q+1} \quad = \quad \int_0^{\infty} J_0^{2q+1}(\psi)\psi d\psi \ , \ J_0(\psi) := \sum_{k=0}^{\infty}\frac{(-1)^{k+1}(x/2)^{2k}}{(k!)^2} \ .$$

Hence, with probability one

$$\frac{meas(x : f_\ell(x) > 0)}{meas(x : f_\ell(x) < 0)} \to 1 \text{ as } \ell \to \infty \ .$$

Nodal Area: the Central Limit Theorem

Theorem

(M and Wigman (2014)) As $\ell \to \infty$

$$d_W\left(\frac{\mathcal{L}_2(A_0(\mathbb{S}^n), f_\ell) - \mathbb{E}\left[\mathcal{L}_2(A_0(\mathbb{S}^n), f_\ell)\right]}{\sqrt{Var\left\{\mathcal{L}_2(A_0(\mathbb{S}^n), f_\ell)\right\}}}, Z\right) = o_{\ell\to\infty}(1) \ ,$$

and hence

$$\frac{\mathcal{L}_2(A_0(\mathbb{S}^n), f_\ell) - \mathbb{E}\left[\mathcal{L}_2(A_0(\mathbb{S}^n), f_\ell)\right]}{\sqrt{Var\left\{\mathcal{L}_2(A_0(\mathbb{S}^n), f_\ell)\right\}}} \to_d N(0,1) \ .$$

Proof

Idea of the proof

$$\mathcal{L}_2(A_0(\mathbb{S}^n), f_\ell) - \mathbb{E}\left[\mathcal{L}_2(A_0(\mathbb{S}^n), f_\ell)\right]$$

$$= \sum_{k=1}^{M} \frac{(-1)^{2k+1}}{(2k+1)!} H_{2k}(u)\phi(u) h_{\ell;2k+1} + R_M \ ,$$

where, as $M \to \infty$,

$$R_M = \sum_{k=M+1}^{\infty} \frac{(-1)^{2k+1}}{(2k+1)!} H_{2k}(u)\phi(u) h_{\ell;2k+1} = o_p(\sqrt{Var\left\{\mathcal{L}_2(A_0(\mathbb{S}^n), f_\ell)\right\}})$$

The proof can be completed by an application of the multivariate Fourth Moment Theorem to the terms $(h_{\ell;3}, ..., h_{\ell;2M+1})$

Nodal Length

Much more attention has been devoted to random nodal lengths, i.e., in our case:

$$Len(f_\ell, \mathbb{S}^2) := 2\mathcal{L}_1(A_0(f_\ell.), \mathbb{S}^2)$$

A groundbreaking paper by Wigman (2010) showed that

$$Var\left[Len(f_\ell, \mathbb{S}^2)\right] = \frac{\log \ell}{32} + O_{\ell\to\infty}(1) \ .$$

Hence, with probability one (compare Yau's conjecture)

$$\frac{Len(f_\ell, \mathbb{S}^2)}{\sqrt{\lambda_\ell}} \to \sqrt{2}\pi \text{ , as } \ell \to \infty \ .$$

The Reduction Principle for Nodal Lines

More recently, it was shown that

Theorem

(M, Rossi and Wigman (2020)) As $\ell \to \infty$

$$Len(\mathbb{S}^2; f_\ell) - \mathbb{E}\left[Len(\mathbb{S}^2; f_\ell)\right] = -\frac{1}{4}\sqrt{\frac{\lambda_\ell}{2}}\frac{1}{4!}h_{\ell;4} + o_p(\sqrt{Var\left\{h_{\ell;4}\right\}}) \ , \tag{3}$$

and hence

$$d_W\left(\frac{Len(\mathbb{S}^2; f_\ell) - \mathbb{E}\left[Len(\mathbb{S}^2; f_\ell)\right]}{\sqrt{Var\left\{Len(\mathbb{S}^2; f_\ell)\right\}}}, \mathcal{N}(0,1)\right) = o_{\ell\to\infty}(1) \ .$$

Hence the nodal lengths is asymptotically proportional to the integral of $H_4(f_\ell)$ (a related result on the torus was given by M, Peccati, Rossi and Wigman (2016); for $\mathbb{R}^2$, see Nourdin, Peccati, Rossi (2019) and Vidotto (2021)).

Correlation between Nodal and Boundary Lengths

The Correlation between Nodal and Boundary Lengths is asymptotically zero, because the former is dominated by the second chaos, the latter by the fourth.
However, subtracting the effect on the boundary length of the random norm $\|f_\ell\|_{L^2(\mathbb{S}^2)}$, one obtains full correlation.
(M and Rossi 2021)

$$\lim_{\ell\to\infty} Corr(Len(\mathbb{S}^2; f_\ell), \mathcal{L}_1(A_u(\mathbb{S}^n), f_\ell)) = 0,$$

$$\lim_{\ell\to\infty} Corr^*(Len(\mathbb{S}^2; f_\ell), \mathcal{L}_1(A_u(\mathbb{S}^n), f_\ell)) = 1 .$$

Corrected Boundary Length

More generally, it can be shown that the fourth-chaos for the boundary length at any level u (possibly different from zero) is asymptotic to

$$\sqrt{\frac{\lambda_\ell}{2}} \frac{\pi}{8} \phi(u) \{2H_4(u) + 4H_4(u) - 3\} \frac{1}{4!} \int_{S^2} H_4(f_\ell(x)) dx$$

Reduction Principle for Total Critical Points

Building on Cammarota and Wigman (2019), it was shown in Cammarota and M (2020) that

$$\mathcal{N}_{-\infty}(\mathbb{S}^2; f_\ell) - \mathbb{E}\left[\mathcal{N}_{-\infty}(\mathbb{S}^2; f_\ell)\right] = -\frac{\lambda_\ell}{2^3 3^2 \sqrt{3}\pi} h_{\ell;4} + o_p(\ell^2 \log \ell) \; ;$$

as a consequence, the nodal lengths and critical points are perfectly correlated in the high-energy limit:

$$\lim_{\ell \to \infty} Corr^2(Len(\mathbb{S}^2; f_\ell), \mathcal{N}_{-\infty}(\mathbb{S}^2; f_\ell)) = 1 \; .$$

An Overview of the Reduction Principles

- At $u \neq 0$, LKC's and Critical Values are dominated by $h_{\ell,2} = \int_{S^2} H_2(f_\ell)dx$
- At $u = 0$, Nodal Lengths and Critical Points are dominated by $h_{\ell,4} = \int_{S^2} H_4(f_\ell)dx$
- At $u \neq 0$, the norm-corrected boundary length is dominated by $h_{\ell,4} = \int_{S^2} H_4(f_\ell)dx$
- At $u = 0$, the Excursion Area is dominated by $h_{\ell,2q+1} = \int_{S^2} H_{2q+1}(f_\ell)dx$, $q = 1, 2, 3, \ldots$.

The Asymptotic Correlation Table

The limiting value of $Corr^2(.,.)$, as $\ell \to \infty$

	$\mathcal{L}_j(u_1)$	$\mathcal{L}_j(u_2)$	$Len(0)$	$Len^*(u)$	$\mathcal{L}_2(0)$	$\mathcal{N}_u$	$\mathcal{N}_{-\infty}$
$\mathcal{L}_j(u_1)$	1	1	0	0	0	1	0
$\mathcal{L}_j(u_2)$	1	1	0	0	0	1	0
$Len(0)$	0	0	1	1	0	0	1
$Len^*(u)$	0	0	1	1	0	0	1
$\mathcal{L}_2(0)$	0	0	0	0	1	0	0
$\mathcal{N}_u$	1	1	0	0	0	1	0
$\mathcal{N}_{-\infty}$	0	0	1	1	0	0	1

Related Works and Conclusions

- Some analogous (but not equivalent) Reduction Principles hold for random eigenfunctions on the torus (*Arithmetic Random Waves*: Rudnick and Wigman (2008), Krishnapur, Kurlberg and Wigman (2013), M, Peccati, Rossi, Wigman (2016), Benatar, M and Wigman (2020), and many others...)
- Related reduction principles hold for random eigenfunctions on $\mathbb{R}^2$ (Berry's Random Waves) (Nourdin, Peccati, Rossi (2019))
- The results on the sphere can be extended to the local behaviour of random spherical harmonics (Todino 2020)
- The results on the zeroes can be extend to random sections of fiber bundles (spin spherical harmonics) (Lerario et al., ongoing)
- Some related Reduction Principles hold for excursion sets of random fields on $\mathbb{S}^2 \times \mathbb{R}$ (M, Rossi and Vidotto (2021)