

The minimality of Sturm-Liouville problems with a boundary condition depending quadratically on the eigenparameter

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First boundary value problems

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- Example: Brachistrone problem $dy = \sqrt{x/(a-x)}dx$ with boundary conditions $y(x_1) = y_1, y(x_2) = y_2$.

The work of Sturm

(1836) The process of heat propagation in an inhomogeneous thin bar

$$g(x) \frac{\partial u}{\partial t} = \frac{\partial \left(k(x) \frac{\partial u}{\partial x} \right)}{\partial x} - n(x)u$$

with initial conditions $u(x, 0) = f(x)$.

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Separation of variables $u(x, t) = X(x)T(t)$.

$$\frac{dT(t)}{dt} = -\lambda T(t), \quad \frac{d(k(x)dX(x)/dx)}{dx} + (\lambda g(x) - n(x))X(x) = 0$$
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Boundary conditions

$$k(0)X'(0) - hX(0) = 0, \quad k(l)X'(l) + HX(l) = 0.$$

The work of Liouville

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- $|u_n(x)| < M/n^2$
- $\sqrt{\lambda_n} \sim \frac{(n-1)\pi}{\gamma} + \frac{P_0}{(n-1)\pi}$

Hyperbolic type PDEs

Partial differential equations of the Hyperbolic type appear when there are processes related to waves. The simplest equation of this type is the *wave equation*:

$$U_{tt} - c^2 U_{xx} = 0$$

Parabolic type PDEs

Equations of Parabolic type appear in problems associated with processes of Heat-transfer and Diffusion. The simplest equation of this type is the *diffusion equation*.

$$U_t - kU_{xx} = 0$$

Elliptic type PDEs

Static processes (waves, heat-transfer, diffusion etc.) are described by equations of Elliptic type. The most familiar equation of this type is *Laplace's equation*:

$$U_{xx} + U_{yy} = 0$$

Fourier Method or Separation of Variables

Standard method of solving these equations is *Fourier Method* or *Separation of Variables*. The main idea behind this method is to assume that the solution can be represented as a product of two functions, first of which depends only on x , and the second one only on t , that is in the form

$$U(x, t) = y(x)g(t)$$

Fourier Method or Separation of Variables

By applying this for example (see [1, p. 108]) to parabolic equation

$$U_t - U_{xx} = 0$$

we obtain

$$y(x)g'(t) - y''(x)g(t) = 0$$

[1] David Logan J. Applied Partial Differential Equations, Springer, 1998.

Fourier Method or Separation of Variables

We can write this equality as

$$\frac{g'(t)}{g(t)} = \frac{y''(x)}{y(x)}$$

Since the left side of this equality is dependent only on t , and the right side is dependent only on x , we can write

$$\frac{g'(t)}{g(t)} = \frac{y''(x)}{y(x)} = -\lambda$$

where λ is a constant.

Example

To demonstrate how this method is applied to the boundary conditions, we take the following example:

$$\begin{cases} U_t - U_{xx} = 0, & 0 < x < \pi, \quad t > 0, \\ U(0, t) = 0, \\ U'_t(\pi, t) = U'_x(\pi, t), \\ U(x, 0) = f(x). \end{cases}$$

Example

The condition

$$U(0, t) = 0$$

implies that

$$y(0)g(t) = 0$$

or just

$$y(0) = 0$$

Example

The condition

$$U'_t(\pi, t) = U'_x(\pi, t)$$

can be written as

$$y(\pi)g'(t) = y'(\pi)g(t)$$

or

$$\frac{y'(\pi)}{y(\pi)} = \frac{g'(t)}{g(t)}.$$

Example

Since we denoted

$$\frac{g'(t)}{g(t)} = -\lambda$$

we obtain

$$\frac{y'(\pi)}{y(\pi)} = -\lambda$$

Example

So, we obtain a Sturm-Liouville problem with boundary conditions depending on eigenvalue parameter

$$\begin{cases} -y''(x) = \lambda y(x), & 0 < x < \pi, \\ y(0) = 0, & y'(\pi) = -\lambda y(\pi). \end{cases}$$

This problem is constructed just to show how the eigenparameter can enter the boundary conditions and does not necessarily represent a physical process. But there are many applicational problems which naturally lead to such problems [2, p.161], [3, p. 96].

[2] Tikhonov A.N. and Samarskii A.A. Equations of mathematical physics, Pergamon Press, 1963.

[3] Aramanovich I.G., Levin V.I. 1969. Equations of mathematical physics (in Russian), Moscow, Nauka, 1964.

Poisson and Fourier

Such boundary value problems can be traced back to the works of Poisson [4]. For justification of the method of separation of variables it is useful to know whether the system of eigenfunctions is complete or not. Such questions were asked after the pioneering works of Fourier [5] who studied trigonometric series of form

$$f(x) = \sum_{n=0}^{\infty} a_{\lambda_n} \cos \lambda_n x + b_{\lambda_n} \sin \lambda_n x \quad \tan \lambda \pi = a \lambda$$

[4] Poisson M. Sur la manière d'exprimer les fonctions par des series de quantités périodiques, et sur l'usage de cette transformation dans la résolution de différents problèmes, Journal de l'école royale polytechnique, XVIII cahier, Vol. XI, Paris (1820) 417-489.

[5] Joseph Fourier, Théorie analytique de la chaleur, Paris, 1822.

Another Example

For example, the eigenvalues satisfying transcendental equation

$$\tan \sqrt{\lambda} = \sqrt{\lambda}$$

are the eigenvalues of the Sturm-Liouville problem

$$\begin{cases} -y''(x) = \lambda y(x), & 0 < x < 1, \\ y'(0) = 0, & y'(1) = -\lambda y(1). \end{cases}$$

So, it is natural to ask whether the system of eigenfunctions

$$\{y_n = \cos \sqrt{\lambda_n} x\} \quad (n \geq 0)$$

forms a basis, say, in $L_p(0,1)$ ($p > 1$) or not.

Minimality

This automatically require the study of minimality of this system. Usually, there is redundancy in the system, and one needs to eliminate some of the eigenfunctions from the system to obtain a minimal system. If some of the eigenvalues are multiple, then one can also introduce associated functions corresponding to the eigenfunctions of these repeated eigenvalues and study more intriguing problem of the basis properties of this extended root space. Since the associated functions are not unique, the choice of the included and deleted functions can affect the basis properties and therefore it is helpful to know necessary and sufficient conditions for completeness and minimality. In the coming slides we will discuss a more general Sturm-Liouville problem and study basis properties of its root functions.

Eigenparameter (quadratically) dependent boundary conditions

$$\begin{cases} -y''(x) = \lambda y(x), & 0 < x < 1, \\ y(0) \cos \beta = y'(0) \sin \beta, & 0 \leq \beta < \pi, \\ y(1) = (a\lambda^2 + b\lambda + c)y'(1). \end{cases}$$

Eigenvalues

(Binding, Brown, Watson, Code 2000's)

- (a) all the eigenvalues are real and simple;
- (b) all the eigenvalues are real and all, except one double eigenvalue, are simple;
- (c) all the eigenvalues are real and all, except one triple eigenvalue, are simple;
- (d) all the eigenvalues are simple and all, except a pair of conjugate nonreal complex eigenvalues, are real.

Eigenfunctions and Associated functions

$$-y''_{k+1} + q(x)y_{k+1} = \lambda_k y_{k+1} + y_k$$

$$y_{k+1}(0) \cos \beta = y'_{k+1}(0) \sin \beta$$

$$y_{k+1}(1) = (a\lambda_k^2 + b\lambda_k + c)y'_{k+1}(1) + (2a\lambda_k + b)y'_k(1);$$

$$-y''_{k+2} + q(x)y_{k+2} = \lambda_k y_{k+2} + y_{k+1}$$

$$y_{k+2}(0) \cos \beta = y'_{k+2}(0) \sin \beta$$

$$y_{k+2}(1) = (a\lambda_k^2 + b\lambda_k + c)y'_{k+2}(1) + \\ + (2a\lambda_k + b)y'_{k+1}(1) + ay'_k(1)$$

1. If all the eigenvalues are simple (real or non-real) then it is a basis in $L_p(0, 1)$.



$y_1(x)$



$y_2(x)$

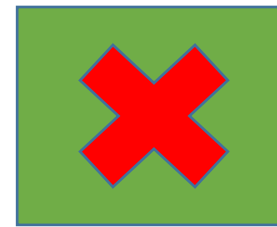
...



$y_i(x)$



...



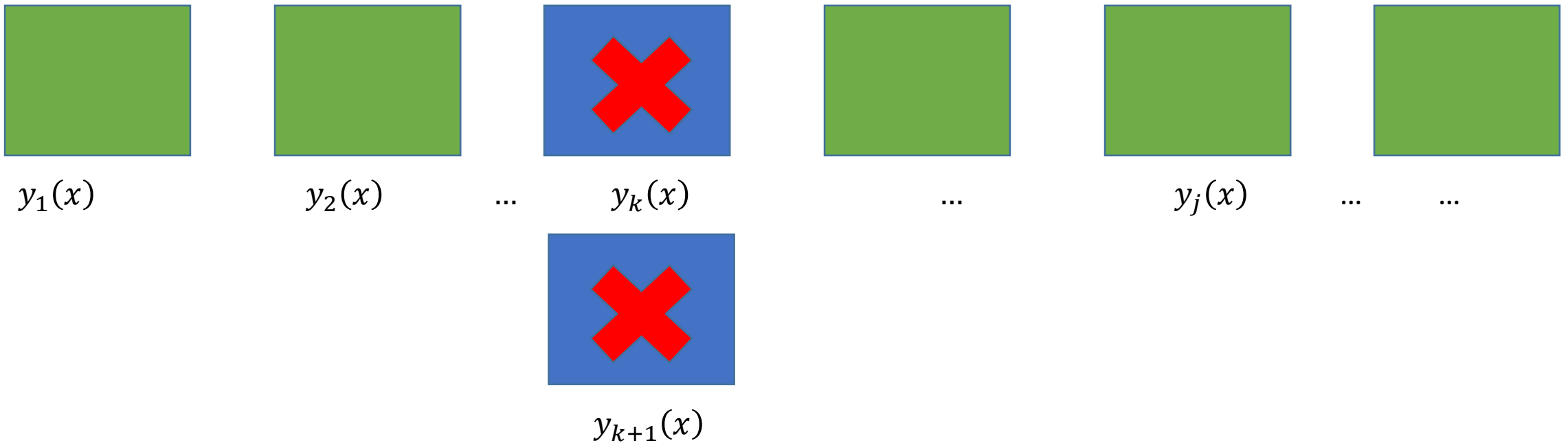
$y_j(x)$

...

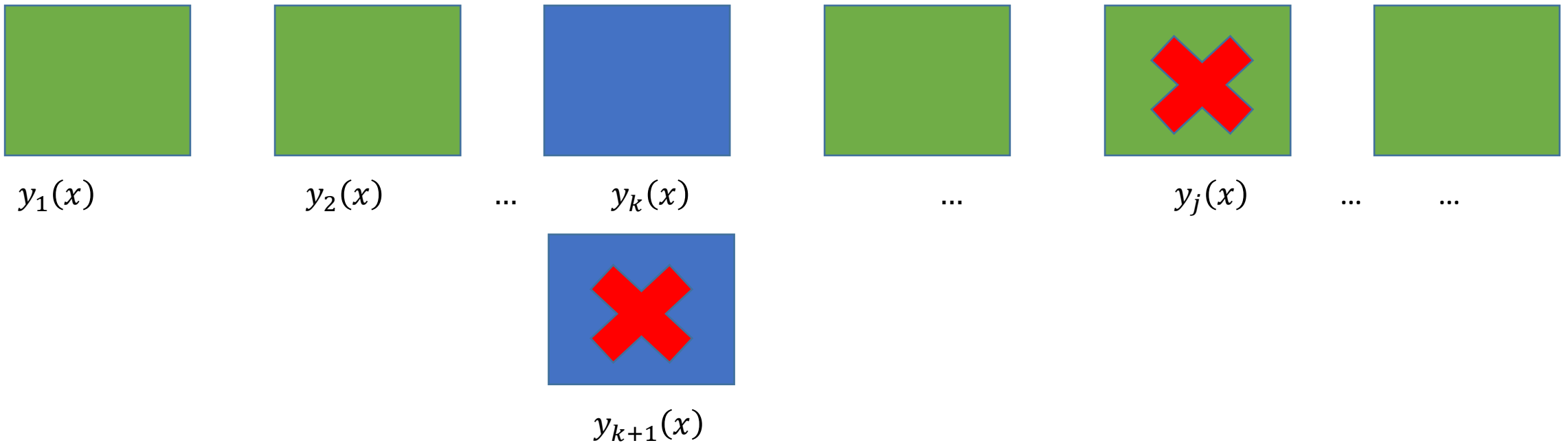


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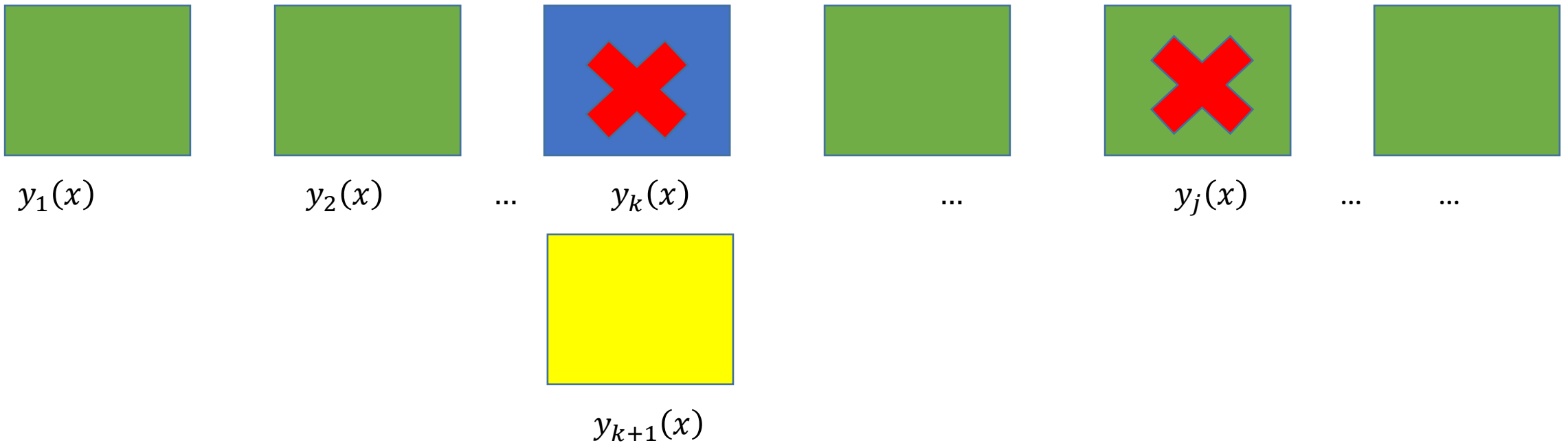
2. If one of the eigenvalues is double then the following system is a basis in $L_p(0, 1)$.



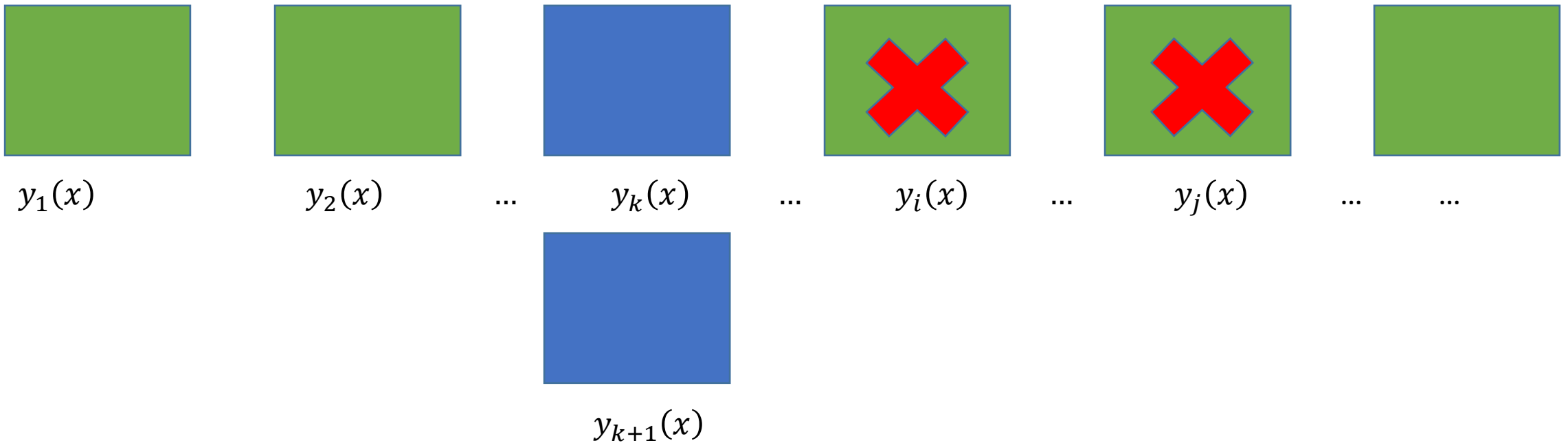
3. If one of the eigenvalues is double then the following system is a basis in $L_p(0, 1)$.



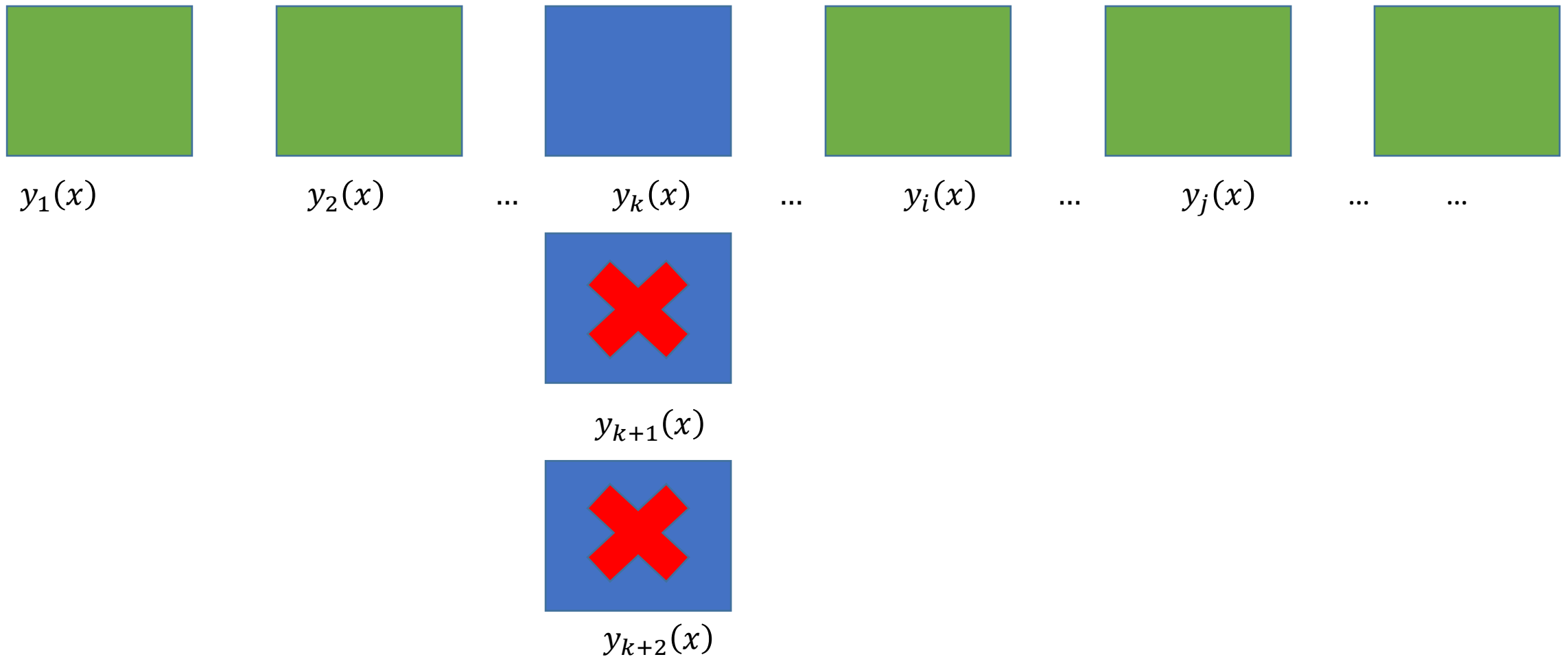
4. If one of the eigenvalues is double then the following system is a basis in $L_p(0, 1)$ under some conditions



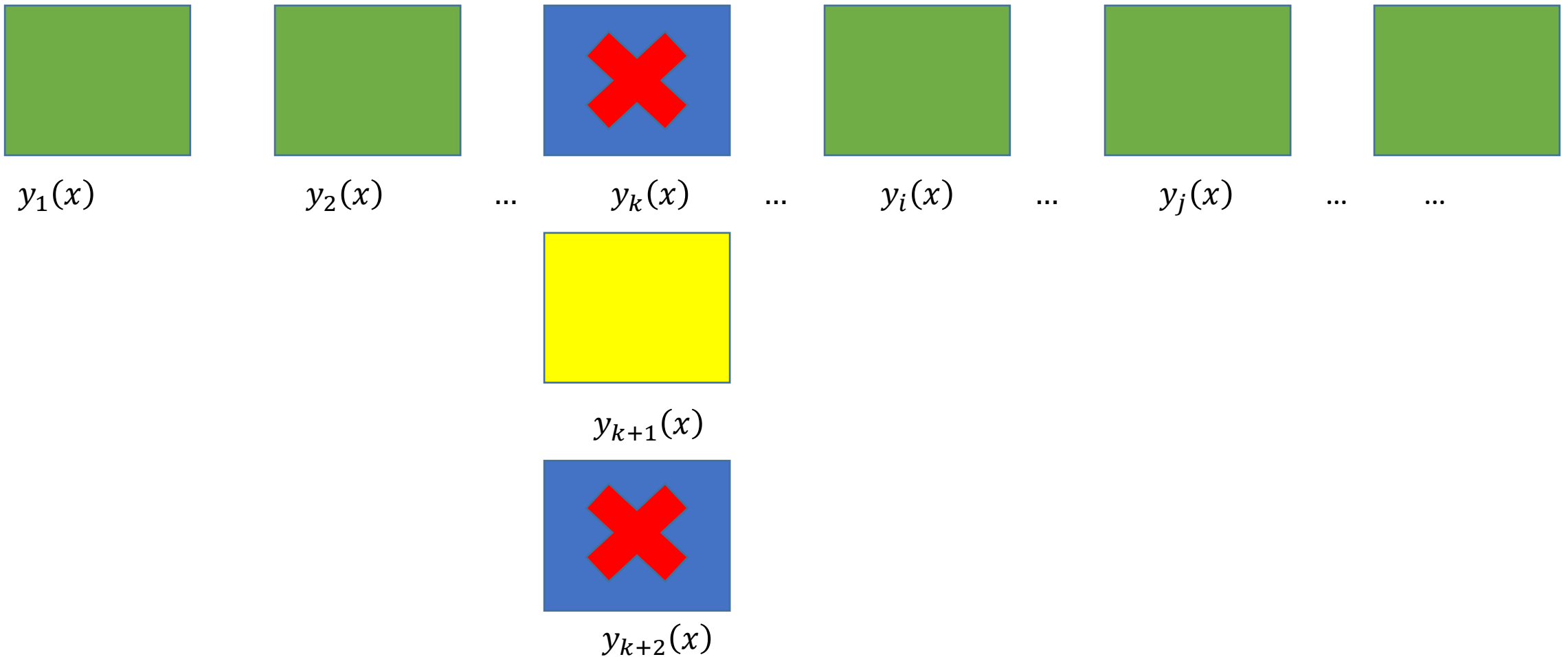
5. If one of the eigenvalues is double then the following system is a basis in $L_p(0, 1)$



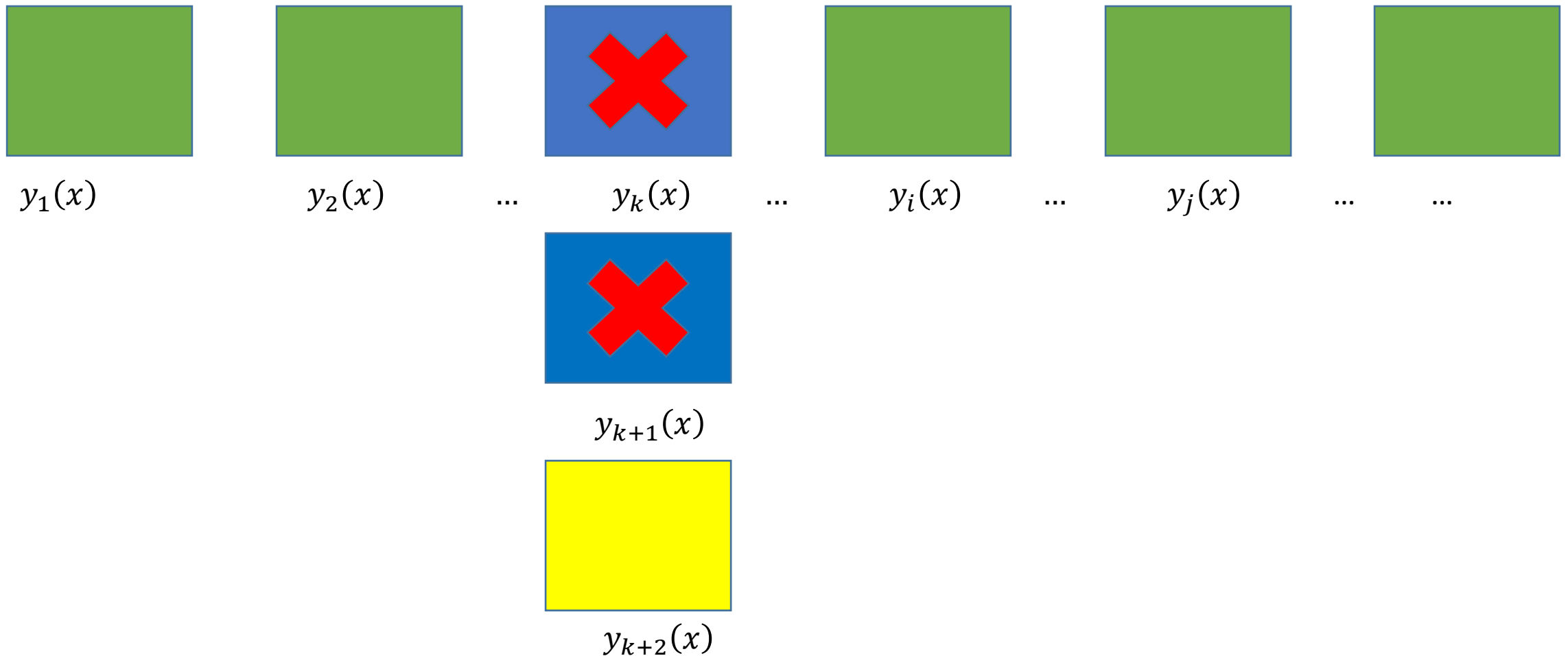
6. If one of the eigenvalues is triple then the following system is a basis in $L_p(0, 1)$.



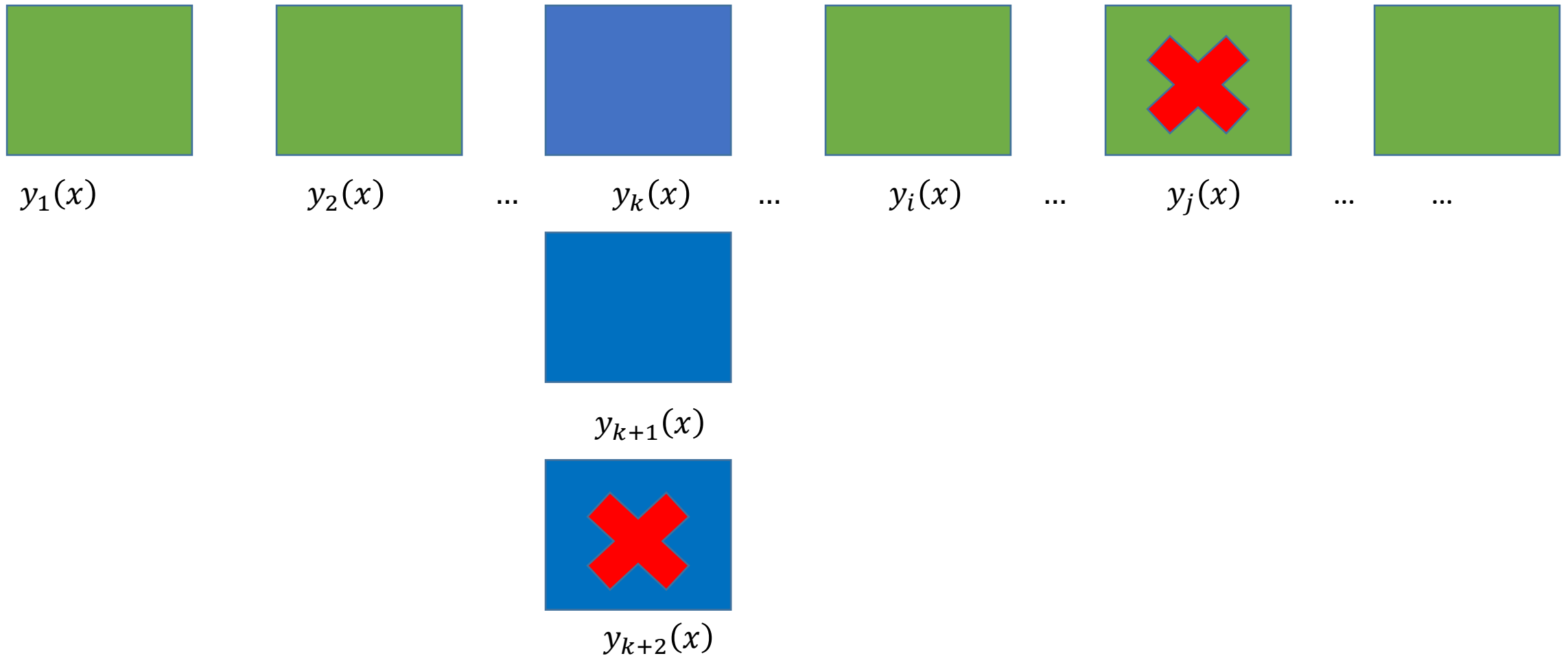
7. If one of the eigenvalues is triple then the following system is a basis in $L_p(0, 1)$ under some conditions



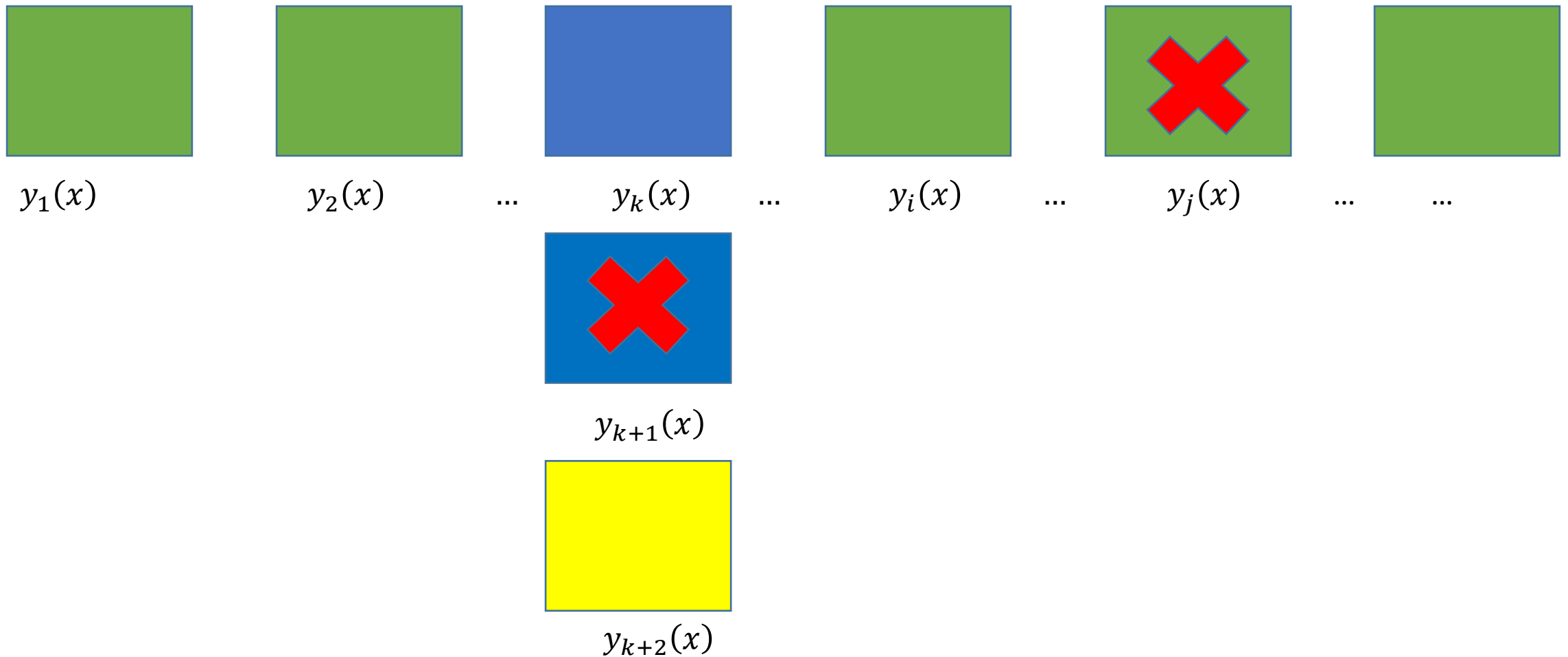
8. If one of the eigenvalues is triple then the following system is a basis in $L_p(0, 1)$ under some conditions



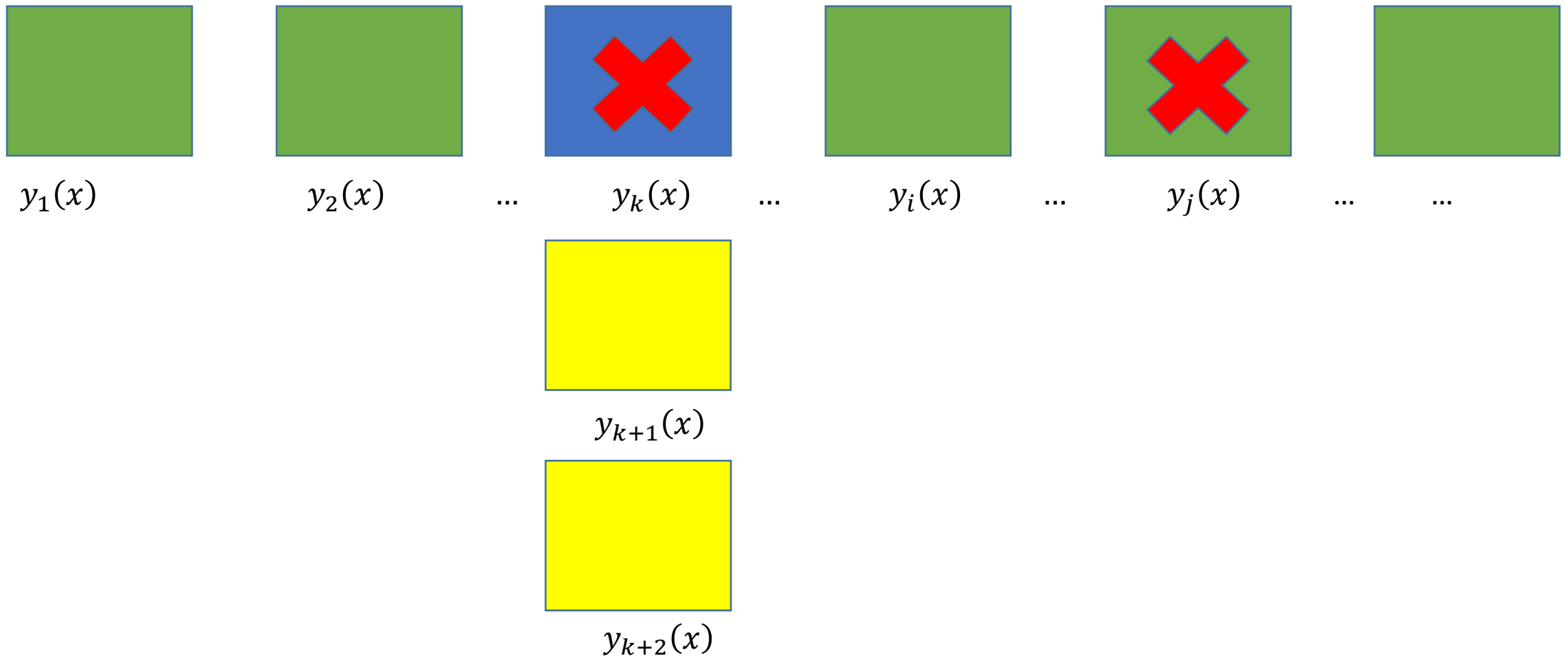
9. If one of the eigenvalues is triple then the following system is a basis in $L_p(0, 1)$



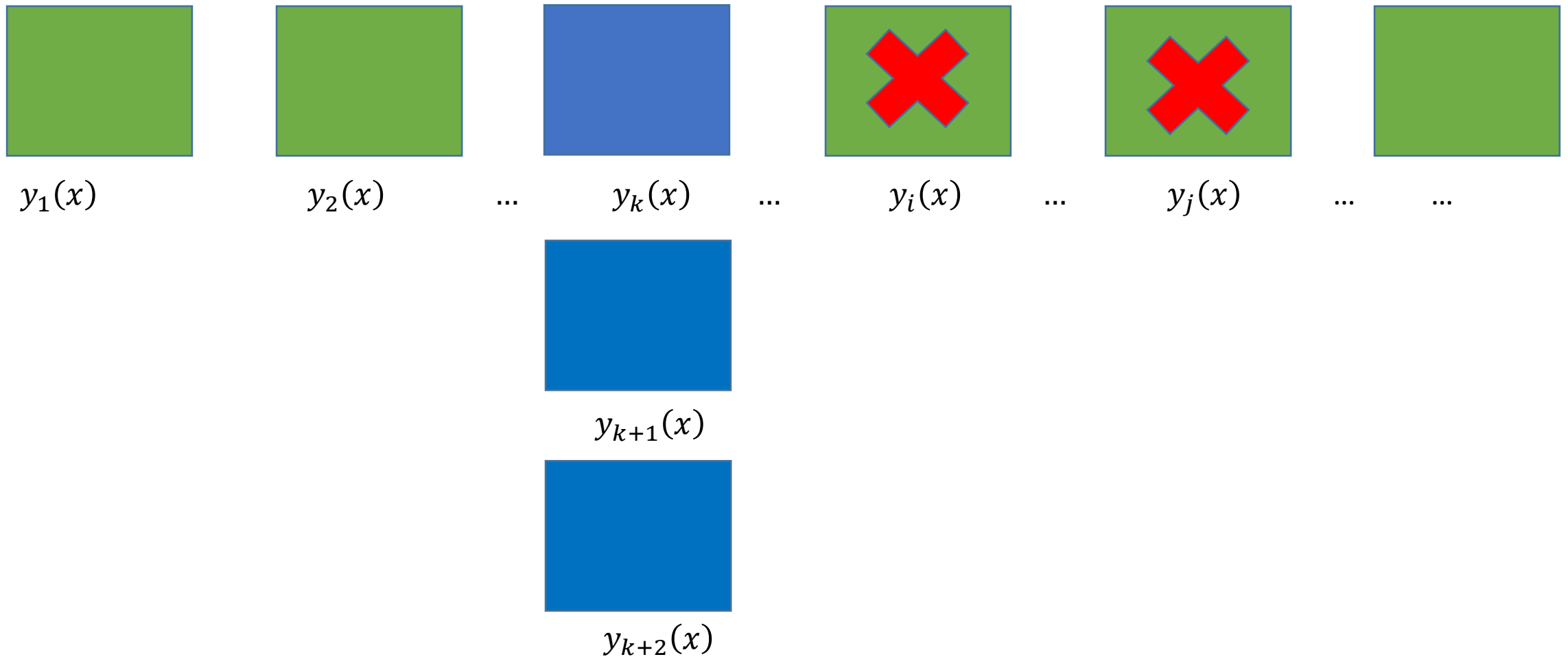
10. If one of the eigenvalues is triple then the following system is a basis in $L_p(0, 1)$ under some conditions



11. If one of the eigenvalues is triple then the following system is a basis in $L_p(0, 1)$ under some conditions



12. If one of the eigenvalues is triple then the following system is a basis in $L_p(0, 1)$ under some conditions



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Thank you!