Resolving sets and identifying codes in finite geometries

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Definitions from finite geometry

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a connected, finite point-line incidence geometry.

$\mathcal{P}$ and $\mathcal{L}$ are two distinct sets, the elements of $\mathcal{P}$ are called points, the elements of $\mathcal{L}$ are called lines. $\mathrm{I} \subset (\mathcal{P} \times \mathcal{L}) \cup (\mathcal{L} \times \mathcal{P})$ is a symmetric relation, called incidence.

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Chain of length h :

$$x_0 \,\mathrm{I}\, x_1 \,\mathrm{I}\, \dots \,\mathrm{I}\, x_h$$

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where $x_i \in \mathcal{P} \cup \mathcal{L}$.
The distance of two elements $d(x, y)$: length of the shortest chain joining them.

Generalized polygons

Definition

Let $n > 1$ be a positive integer. $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is called a generalized n-gon if it satisfies the following axioms.

- **Gn1.** $d(x, y) \leq n \;\forall\; x, y \in \mathcal{P} \cup \mathcal{L}$.
- **Gn2.** *If* $d(x, y) = k < n$ *then* $\exists!$ *a chain of length* k *joining* x *and* y.
- **Gn3.** $\forall\; x \in \mathcal{P} \cup \mathcal{L} \;\exists\; y \in \mathcal{P} \cup \mathcal{L}$ *such that* $d(x, y) = n$.

A generalized n-gon is thick if each of its elements has at least three neighbours.

Some elementary observations

- For the graph-theorists: the point-line incidence graph (*Levi graph*) of a generalized n-gon is a connected bipartite graph of diameter n and girth $2n$.
- The simplest example: n-gon in the euclidean plane.

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- The simplest example: n-gon in the euclidean plane.
- The dual of a generalized n-gon is also a generalized n-gon.
- The distance of two points or two lines is even. The distance of a point and a line is odd.

Existence

Theorem (Feit, Higman, 1964)

Finite thick generalized n-gons exist if and only if $n = 2, 3, 4, 6$ *and* 8.

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$n = 3$: projective planes

$n = 4$: generalized quadrangles

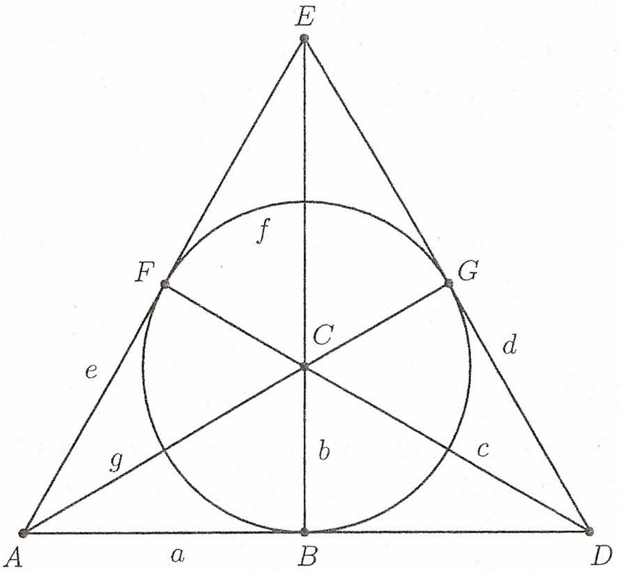
$n = 3$, projective planes

Definition

$\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ *is called a projective plane if it satisfies the following axioms.*

- **P1.** *For any two distinct points there is a unique line joining them.*
- **P2.** *For any two distinct lines there is a unique point of intersection.*
- **P3.** *Each line is incident with at least three points and each point is incident with at least three lines.*

The Fano plane



Affine and biaffine planes

Let Π_q be a projective planeof order q.. If we delete a line ℓ of Π and all points on ℓ, then we get an affine plane of order q.

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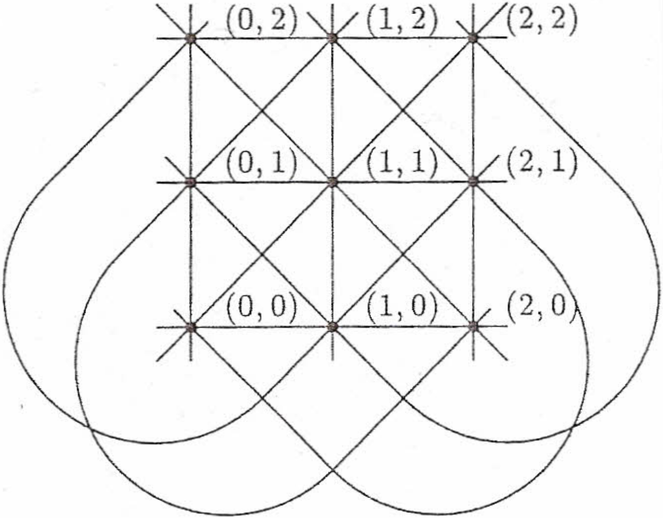
We denote by $\mathrm{BG}(2, q)$ the Desarguesian biaffine plane; that is, the point-line incidence structure obtained from $\mathrm{AG}(2, q)$ by removing the lines of an arbitrary parallel class.

AG(2, 3)



$n = 4$, GQ-s

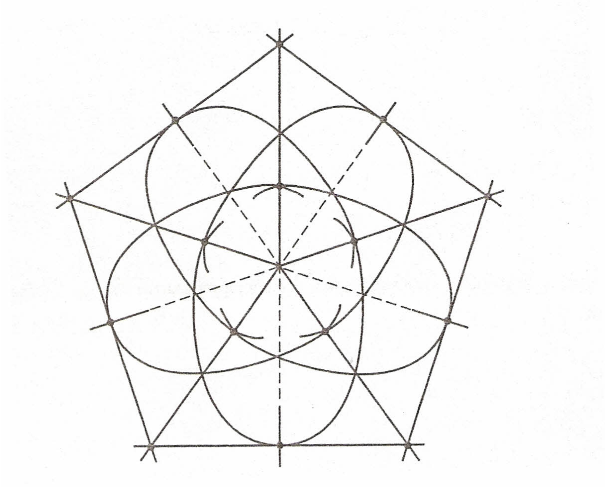
Theorem

Let $\mathcal{Q}$ be a finite, thick generalized quadrangle of order (s, t). Then

1. *For each non-incident point-line pair (P, e) $\exists!$ a point-line pair (R, f) such that $P \,\mathrm{I}\, f \,\mathrm{I}\, R \,\mathrm{I}\, e$,*
2. *$\mathcal{Q}$ contains $(s+1)(st+1)$ points,*
3. *$\mathcal{Q}$ contains $(t+1)(st+1)$ lines.*

The smallest nontrivial example

$s = t = 2$, 15 points, 15 lines



Definitions from graph theory

Definition

Let $\Gamma = (V, E)$ *be a finite, simple, undirected graph. A vertex* $v \in V$ *is resolved by* $S = \{v_1, \ldots, v_n\} \subset V$ *if the list of distances* $(d(v, v_1), d(v, v_2), \ldots, d(v, v_n))$ *is unique.* S *is a resolving set for* Γ *if it resolves all the elements of* V.
The metric dimension of Γ*, denoted* $\mu(\Gamma)$*, is the smallest size of a resolving set for* Γ.

Motivation

The *base size of a permutation group* is the smallest number of points whose stabilizer is the identity. The study of base size dates back more than 40 years.
The *base size* of Γ, denoted $b(\Gamma)$, is the base size of its automorphism group.

A resolving set for Γ is a base for $\mathrm{Aut}(\Gamma)$, so the metric dimension of a graph gives an upper bound on its base size.

The *dimension jump* of Γ, denoted $\delta(\Gamma)$, is the difference of its metric dimension and its base size

$$\delta(\Gamma) = \mu(\Gamma) - b(\Gamma).$$

The best known general bound

Theorem (Hernando et al., 2010)

If Γ *has* n *vertices, its diameter is* d *and* $\mu(\Gamma) = k$ *then*

$$n \leq \left(\left\lfloor \frac{2d}{3} \right\rfloor + 1 \right)^k + k \sum_{i=1}^{\lceil d/3 \rceil} (2i-1)^{k-1}.$$

Projective planes

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Theorem (Héger and Takáts, 2012)

If $q \geq 23$*, then the metric dimension of a finite projective plane of order q is 4q-4.*
There are 32 types of resolving sets of size 4q-4.

Affine and biaffine planes

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Theorem (DB-TH-GyK-MT, 2018)

- *If* $q \geq 4$ *then the metric dimension of* $\mathcal{A}_q$ *is at most* $3q-4$.
- *If* $q \geq 11$ *then the metric dimension of* $\mathrm{AG}(2,q)$ *is* $3q-4$.

Theorem (DB-TH-GyK-MT, 2018)

Let Γ_q *be the incidence graph of* $\mathrm{BG}(2,q)$. *Then*

$$8q/3-8 \leq \mu(\Gamma_q) \leq 3q-6.$$

The proof consists of several steps by showing a connection between small resolving sets of $\mathrm{BG}(2,q)$ and blocking sets of $\mathrm{PG}(2,q)$.

$W(q)$

Theorem (DB-THGyK-MT, 2018)

Let Γ_q be the incidence graph of $W(q)$. Then

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The proof is based on the geometric properties of the null-polarity from which $W(q)$ arises.

The Levi graph of $\mathrm{PG}(n, q)$

Let $\Gamma_{P,\mathcal{H}}(n, q)$ be the point-hyperplane incidence graph of the finite projective space $\mathrm{PG}(n, q)$. The two sets of vertices of this bipartite graph correspond to the points and hyperplanes of $\mathrm{PG}(n, q)$, respectively, and there is an edge between two vertices if and only if the corresponding point is in the corresponding hyperplane.

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Theorem (DB-GyK-SM-FP, 2020)

If q is large enough then the size of any resolving set in $\Gamma_{P,\mathcal{H}}(n, q)$ is at least

$$r = 2nq - 2\frac{n^{n-1}}{(n-1)!}.$$

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$$r = 2nq - 2\frac{n^{n-1}}{(n-1)!}.$$

In the planar case our bound is tight. By the theorem of Héger and Takáts, the metric dimension of the point-line incidence graph of a projective plane of order q is $4q - 4$.

Restricted resolving sets

Theorem (Fancsali and Sziklai, 2014)

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Theorem

The graph $\Gamma_{P,\mathcal{H}}(3, q)$ has a resolving set of size $8q$.

One-sheeted hyperboloid

Higgledy-piggledy lines

Definition

In $\mathrm{PG}(n, q)$ a set $\mathcal{L}$ of lines is called a *higgledy-piggledy arrangement* if no $(n-2)$-dimensional subspace meets each element of $\mathcal{L}$.

Higgledy-piggledy lines

Theorem (Fancsali and Sziklai, 2014)

If $q = p^r$, $p > n$ *and* $q \geq 2n - 1$ *then the graph* $\Gamma_{P,\mathcal{H}}(n, q)$ *has a resolving set of size* $(4n - 2)q$.

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Theorem (DB-GyK-SM-FP, 2020)

The graph $\Gamma_{P,\mathcal{H}}(n, q)$ has a resolving set of size $(n^2 + n - 4)q$.

The main theorem

Theorem (DB-GyK-SM-FP, 2020)

Let $p \neq 2, 3$ and $q = p^r > 36086$. Then the graph $\Gamma_{P,\mathcal{H}}(4, q)$ has a resolving set of size $12q$.

The main theorem

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The proof is based on the following propositiom. Its proof uses Grassmann coordinates and algebraic geometry.

Proposition

In $\mathrm{PG}(4, p^r)$ *if* $p \neq 2$, $p \neq 3$ *and* $p^r > 36086$, *then there exists* $\alpha \in \mathrm{GF}(p^r)$ *such that there is no plane which intersects each of the six lines joining the pairs of points*

$$\begin{array}{ccc}
(1:0:0:0:0) & \textit{and} & (0:1:1:0:0), \\
(0:1:0:0:0) & \textit{and} & (0:0:1:1:0), \\
(0:0:1:0:0) & \textit{and} & (0:0:0:1:1), \\
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\end{array}$$

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- *A subset of vertices $D \subset V$ is a dominating set if each vertex is either in D or adjacent to a vertex in D.*

Identifying codes

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- *A subset of vertices $D \subset V$ is a dominating set if each vertex is either in D or adjacent to a vertex in D.*
- *A vertex s separates u and v if exactly one of u and v is in $N[s]$. A subset of vertices $S \subset V$ is a separating set if it separates every pair of vertices of G.*

Identifying codes

Definition

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- *A subset of vertices* $D \subset V$ *is a dominating set if each vertex is either in* D *or adjacent to a vertex in* D.
- *A vertex* s *separates* u *and* v *if exactly one of* u *and* v *is in* $N[s]$. *A subset of vertices* $S \subset V$ *is a separating set if it separates every pair of vertices of* G.
- *A subset of vertices* $C \subset V$ *is an identifying code for* V *if it is both a dominating and separating set.*

In other words, if C is an identifying code then the set $C \cap N(v)$ is non-empty and uniquely determines v.

Genaral bounds

The size of a minimal identifying code for G is denoted by $\gamma^{\mathrm{id}}(G)$. The best known general bound was proven by *Karpovsky, Chakrabarty* and *Levitin*:

Theorem (Karpovsky, Chakrabarty, Levitin, 1998)

If G is a twin-free graph with at least one edge then

$$\log_2(|V|+1) \leq \gamma^{\mathrm{id}}(G) \leq |V|-1.$$

Motivation

Several pracical applications:

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Connection with resolving sets:

Theorem (Gravier et al. 2015)

If G has diameter 2 then

$$\mu(G) \leq \gamma^{\mathrm{id}}(G) \leq 2\mu(G) + 2.$$

Bounds for adjacency graphs of GQs

Better bounds were proved for vertex-transitive and strongly regular graphs by *Gravier et al.*

Theorem (Gravier et al., 2015)

Let G be a vertex-transitive GQ of order (s, t) with $s > 1$ and $t > 1$. If n denotes the number of vertices in G_{GQ} then

$$2^{-5/4} \cdot n^{1/4} \leq \gamma^{\mathrm{id}}(G_{GQ}) \leq 2 \cdot n^{2/5}.$$

Bounds for adjacency graphs of GQs

An improved general lower bound:

Theorem (TH-GyK-AN-LS, 2021+)

Any identifying code of a GQ of order (s,t) *has size at least* $3\min\{s,t\}-4$.

Bounds for adjacency graphs of GQs

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Theorem (TH-GyK-AN-LS, 2021+)

Any identifying code of a GQ of order (s, t) has size at least $3\min\{s, t\} - 4$.

In particular:

- $\gamma^{\mathrm{id}}(T_2^*(O)^D) \geq 3q - 1$,
- $\gamma^{\mathrm{id}}(Q^-(5, q)) \geq 3q + 3$,
- $\gamma^{\mathrm{id}}(H(3, q^2)) \geq 3q^2 - 8q + 35$,
- $\gamma^{\mathrm{id}}(H(4, q^2)) \geq 3q^2 + 3$,
- $\gamma^{\mathrm{id}}(H(4, q^2)^D) \geq 3q^3 - 8q + 1$.

Upper bounds

The constructions are based on the geometric properties of conics, quadrics and Hermitian surfaces.

Theorem (TH-GyK-AN-LS, 2021+)

- $\gamma^{\mathrm{id}}(T_2^*(O)^D) \leq 6q+4$,
- $\gamma^{\mathrm{id}}(W(q)) \leq 4q+1$ *if* q *is odd,*
- $\gamma^{\mathrm{id}}(H(3,q^2)) \leq 5q^2-2$ *if* $q>2$,
- $\gamma^{\mathrm{id}}(H(4,q^2)) \leq 8q^2+q-7$ *if* $q>2$,

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THANKS FOR YOUR ATTENTION!

HVALA ZA VAŠO POZORNOST!