Stokes Equations In An Infinite Strip With a Hole And transmission Conditions

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Let $0 < a_i < b_i < l_i$, $i = 1, 2$ and $S = (0, l_1) \times (0, l_2)$, $\widetilde{S} = (a_1, b_1) \times (a_2, b_2)$. Let also $Y \subset S \times]-1, 1[$ be a convex open set with smooth boundary ∂Y. Let Λ be the infinite vertical domain in $\mathbf{R}^3$ defined by

$$\Lambda = \big(S \times]-\infty, +\infty[\big) \setminus \overline{Y}.$$

We define the following subsets in Λ:

$$\begin{array}{rcl} \Lambda_- & = & \left\{y = (y', y_3) \in \mathbf{R}^3 \ ; \ y' \in S \ , \ y_3 < -1\right\}, \\ \Omega & = & \big(S \times]-\infty, +\infty[\big) \setminus \overline{Y}, \\ \Lambda_+ & = & \left\{y = (y', y_3) \in \mathbf{R}^3 \ ; \ y' \in \widetilde{S} \ , \ y_3 > 1\right\}, \\ \Gamma_- & = & \left\{y = (y', y_3) \in \mathbf{R}^3 \ ; \ y' \in S \ , \ y_3 = -1\right\}, \\ \Gamma_- & = & \left\{y = (y', y_3) \in \mathbf{R}^3 \ ; \ y' \in S \ , \ y_3 = 1\right\}, \end{array}$$

where we denoted $y' = (y_1, y_2)$. Then we can decompose Λ as follows:

$$\Lambda = \Lambda_- \cup \overline{\Omega} \cup \Lambda_+.$$

We seek a couple (u, p) defined in Λ as

$$u(x) = \begin{cases} u_-(x), & x \in \Lambda_- \\ u_0(x), & x \in \Omega \\ u_+(x), & x \in \Lambda_+ \end{cases} \quad \text{and} \quad p(x) = \begin{cases} p_-(x), & x \in \Lambda_- \\ p_0(x), & x \in \Omega \\ p_+(x), & x \in \Lambda_+ \end{cases}$$

where the pairs (u_-, p_-), (u_0, p_0) and (u_+, p_+) satisfy the following system

$$(\mathrm{S}) \left\{ \begin{array}{rcll} -\nu \Delta u_\pm + \nabla p_\pm & = & 0 & \text{in } \Lambda_\pm, \\ -\nu \Delta u_0 + \nabla p_0 & = & 0 & \text{in } \Omega, \\ \nabla \cdot u_\pm & = & 0 & \text{in } \Lambda_\pm, \\ \nabla \cdot u_0 & = & 0 & \text{in } \Omega, \\ u_0 & = & 0 & \text{on } \partial Y \\ \sigma(u_-, p_-) \cdot \mathrm{n} & = & \sigma(u_0, p_0) \cdot \mathrm{n} + \nu\, g & \text{on } \Gamma_-, \\ \sigma(u_0, p_0) \cdot \mathrm{n} & = & \sigma(u_+, p_+) \cdot \mathrm{n} + \nu\, h & \text{on } \Gamma_+, \end{array} \right.$$

with (u_-, p_-) and (u_+, p_+) are periodic with respect to y_1 and y_2, with periods l_1 and l_2. Here $\nu > 0$ is the viscosity parameter and n is the unit normal vector on Γ_- (resp. Γ_+) external to Λ_- (resp. Ω), i.e. $\mathrm{n} = (0, 0, 1)$. The vector functions $g = (g', 0)$ and $h = (h', 0)$ are supposed to be given in suitable function spaces.

We study the existence and uniqueness of a solution (u, p) to the system (S) which decays exponentially fast, as well as all its derivatives, as $y_3 \to \pm\infty$.

The main result of this work is the following:

Theorem: Suppose that

$$g \in (H^{-1/2}_{per}(\Gamma_-))^3, \quad g_3 = 0, \quad h \in (H^{-1/2}_{per}(\Gamma_+))^3, \quad h_3 = 0.$$

There exists a unique solution of the system (S) (up to an additive constant for the pressures) satisfying

$$u \in \left(H^1_{per,loc}(\Lambda)\right)^3, \; \nabla u_- \in \left(L^2(\Lambda)\right)^9, \; p_- \in L^2_{per,loc}(\Lambda).$$

Moreover, let $\delta > 1$ and let $\beta_\pm$ be the mean of the velocity over cross sections of $\Lambda_\pm$, i.e.

$$\beta_- = \frac{1}{|S|} \int_S u_-(y', -\delta)\, dy', \quad \beta_+ = \frac{1}{|S|} \int_S u_+(y', \delta)\, dy'.$$

The following decay estimates hold:

- for any $\alpha \in \mathbf{N}^3$, $y' \in S$, $y_3 \leq -\delta$,

$$|\partial^\alpha (u - \beta_-)(y', y_3)| + |\partial^\alpha p(y', y_3)| \leq C(\delta, \alpha)\, \|g\|_{\left(H^{-1/2}(\Gamma_-)\right)^3} \exp\,(c\, y_3);$$

- for any $\alpha \in \mathbf{N}^3$, $y' \in \widetilde{S}$, $y_3 \geq \delta$,

$$|\partial^\alpha (u - \beta_+)| + |\partial^\alpha p(y', y_3)| \leq C(\delta, \alpha)\, \|h\|_{\left(H^{-1/2}(\Gamma_+)\right)^3} \exp\,(-c\, y_3),$$

where $c > 0$ is a constant independent of the data and $C(\delta, \alpha)$ is a constant depending only on δ and α. The subscript "per" denotes periodic Sobolev Spaces.

This work answers a question addressed to the author by G. Panasenko. It will be be used in a forthcoming work to build boundary layers correctors in an homogenization framework.