Jordan schemes

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Association schemes (also called homogeneous coherent configurations) first appeared in Statistics in relation to the design of experiments, due to the efforts of Bose and his collaborators.

These systems of binary relations defined on the same set, color graphs in other terms, give rise to certain matrix algebras, not necessarily commutative. In this way a bridge is formed between combinatorics and algebra, in particular with permutation groups.

In statistial applications we want the relations to be symmetric, while the product of symmetric matrices is not symmetric in general. Therefore Bailey and Cameron, following Shah (from the same school as Bose) suggested to replace the matrix product AB by the Jordan product $A * B = (AB + BA)/2$, which is commutative but not necessarily associative. The resulting structures are called Jordan schemes.

Given any association scheme, its symmetrization is a Jordan scheme. This led Peter Cameron to the following question: Do all Jordan schemes arise in this way, or do there exist "proper" Jordan schemes?

We gave a positive answer to this question by constructing a first proper Jordan scheme on 15 points using so-called Siamese color graphs, investigated earlier by our group. First elements of the theory of these structures were established; a few infinite classes of proper Jordan schemes were discovered. Moreover an efficient computational criterion for recognizing proper Jordan schemes is given, based on the classical Weisfeiler-Leman stabilization.

The current talk mainly focuses on the computer search for proper Jordan schemes, which is based on algorithmic ideas of Hanaki and Miyamoto, who enumerated all small association schemes. It turns out that the initial example is indeed the smallest. Besides it, up to isomorphism three more examples on less than 20 points were discovered. Each such small example of orders 15, 16 and 18 can be constructed from a suitable association scheme by a certain switching operation.

For more on the background of Jordan schemes see Cameron's blog: https://cameroncounts.wordpress.com/2019/06/28/proper-jordan-schemes-exist/

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