Semidefinite relaxations in non-convex spaces

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In analogy with Lassere's and Parillo's hierarchies of semidefinite relaxations for polynomial optimization problems, a semidefinite hierarchy exisits for non-commutative polynomial optimization problems of the form

$$p^* = \begin{array}{cc} \min\limits_{(\mathcal{H},\phi,X)} & \langle \phi, p(X)\phi \rangle \\ \text{s.t.} & q_i(X) \succeq 0 \end{array},$$

where ϕ is a normalized vector in the Hilbert space $\mathcal{H}$, $X = (X_1, \ldots, X_n)$ are the non-commuting variables in the problem, and $p(X)$ and $q_i(X)$ are polynomials in the variables X. This hierarchy, known as the Navascués-Pironio-Acín hierarchy, has encountered important applications in physics, where it has become a central tool in quantum information theory and in the certification of quantum phenomena.

Its success has motivated its application, within quantum information theory, in more complex scenarios where the relevant search spaces are not convex, and thus the characterizing constraints are in conflict with semidefinite formulations. The paradigmatic example is optimizing over probability distributions obtained by parties performing measurements on quantum systems where not all parties receive a share from every system available. The most illustrating consequence is that, in certain situations, marginalization over a selected number of parties makes the resulting probability distribution to factorize.

In this talk we address the problem of non-commutative polynomial optimization in non-convex search spaces and present two means of providing monotonically increasing lower bounds on its solution, that retain the characteristic that the newly formulated problems remain being semidefinite programs. The first directly addresses factorization-type constraints by adding auxiliary commuting variables that allow to encode (relaxations of) polynomial trace constraints, while the second considers multiple copies of the problem variables and constrains them by requiring the satisfaction of invariance under suitable permutations. In addition to presenting the methods and exemplifying their applicability in simple situations, we highlight the fundamental questions that remain open, mostly regarding to their convergence.