Solving General Itô-Process Hitting-Time Problems with General Moving Boundaries

Martin Nilsson
RISE Research Institutes of Sweden
from.ecm@drnil.com

A spectral method for solving both first-passage time and first-exit time problems for general Itô processes is presented. The method handles general moving (*i.e.*, time-variable) boundaries including discontinuities. Results from the application to neuron modelling are given.

The method is built upon the idea of first expressing the problem as a Fokker–Planck equation over a non-rectangular strip. The Fokker–Planck equation is then spectrally reduced to a small set of ordinary differential equations which can be solved easily and quickly by standard solvers.

The reduction is the key step of the method. It observes that the right hand side of the Fokker–Planck equation

$$\frac{\partial}{\partial t}p = L\,p$$

can be used as the left-hand side of the Sturm-Liouville system

$$L\,p = -\lambda\,p,$$

which generates a moving orthogonal family of eigenfunctions. Fourier-transforming the Fokker–Planck equation in terms of these eigenfunctions produces a series of simple ordinary differential equations for the Fourier coefficients of the solution. For example, in the time-homogeneous first-passage time case, these equations can be expressed as

$$\frac{d\omega}{dt} = \left(C\,\frac{da}{dt} - \Lambda\right)\omega,$$

where ω is the vector of Fourier coefficients, a is the moving boundary, Λ is the diagonal matrix of the moving Sturm–Liouville eigenvalues λ_k, and C is the skew-symmetric matrix defined by

$$[C]_{km} = \frac{1}{\lambda_k - \lambda_m}\sqrt{\frac{\partial\lambda_k}{\partial a}\frac{\partial\lambda_m}{\partial a}}$$

for $k \neq m$.