

## The Hopf-Oleinik Lemma for the divergence-type equations

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The Hopf-Oleinik lemma, known also as the “normal derivative lemma”, is one of the important tools in qualitative analysis of partial differential equations.

This lemma states that *a supersolution of a partial differential equation with a minimum value at a boundary point, must increase linearly away from its boundary minimum provided the boundary is smooth enough.*

A major part of all known results on the normal derivative lemma concerns equations with nondivergence structure and strong solutions (see [1] and [2] for some recent results and the comprehensive historical review). The case of the divergence-type equations is less studied. It is well known that the normal derivative lemma fails for uniformly elliptic equations in divergence form with bounded and even continuous leading coefficients. Thus, one has to require more smoothness of the leading coefficients.

The first result for elliptic equations with divergence structure was proved by R. Finn and D. Gilbarg (1957), who considered 2D bounded domains with  $C^{1,\alpha}$ -regular boundary, the Hölder continuous leading coefficients and continuous lower order coefficients. In our knowledge, the best result preceding our one was established by V. Kozlov and N. Kuznetsov (2018), who considered  $n$ -dimensional bounded  $C^{1,\alpha}$ -domains ( $n \geq 3$ ) for equations with the lower-order coefficients belonging to the Lebesgue space  $L^q$ ,  $q > n$  and the same leading coefficients as before.

For the parabolic divergence-type equations we do not know such results. However, the normal derivative lemma for parabolic equations can be extracted from the lower bound estimates of the Green function for the corresponding operator.

We present several versions of the Hopf-Oleinik lemma for general elliptic and parabolic equations in divergence form under the sharp requirements on the coefficients of equations and on the boundary of a domain. All our as-

sumptions are significantly weakened in comparison with the previous works. In fact, our requirements are close to the necessary ones.

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## References

- [1] A.I. Nazarov, *A centennial of the Zaremba-Hopf-Oleinik lemma*, SIAM J. Math. Anal. **44** (2012), no. 1, 437–453.
- [2] D.E. Apushkinskaya, A.I. Nazarov, *A counterexample to the Hopf-Oleinik lemma (elliptic case)*, Anal. PDE **9** (2016), no. 2, 439–458.
- [3] D.E. Apushkinskaya, A.I. Nazarov, *On the Boundary Point Principle for divergence-type equations*, Rend. Lincei Mat. Appl. **30** (2019), 677–699.