On Muirhead and Schur inequalities

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It is considered three and four term generalizations of the well-known Muirhead inequality. Let positive integers n and m be given, $n \geqslant 2,\ m \geqslant 1$. A vector $\boldsymbol{\alpha} = (\alpha_1,\, \alpha_2, \ldots,\, \alpha_n)$ with nonnegative integer components is called a power if $\alpha_1 \geqslant\ \alpha_2 \geqslant \cdots \geqslant\ \alpha_n$ and $\alpha_1 +\ \alpha_2 + \cdots +\ \alpha_n = m$. The set of all the powers will be denoted by $A(n, m)$. Then

$$\mu_{\boldsymbol{\alpha}}\left(\boldsymbol{x}\right) = \frac{1}{n!} \sum_{\sigma \in S_n} x_{\sigma_1}^{\alpha_1} x_{\sigma_2}^{\alpha_2} \ldots\, x_{\sigma_n}^{\alpha_n}$$

is an elementary homogeneous symmetric polynomial of the vector $\boldsymbol{x} = (x_1,\, x_2, \ldots,\, x_n) \in \mathbb{R}_+^n$ (where S_n is a family of all permutations of the set $\{1,\, 2, \ldots,\, n\}$).

The family $A(n, m)$ will be considered with the order $\boldsymbol{\alpha} \leqslant \boldsymbol{\beta}$, defined by the relations

$$\begin{aligned} &\alpha_1 \leqslant \beta_1, \\ &\alpha_1 + \alpha_2 \leqslant \beta_1 + \beta_2, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} \leqslant \beta_1 + \beta_2 + \cdots + \beta_{n-1}. \end{aligned}$$

The following statements hold:
a) $\left[\forall \boldsymbol{x} \in \mathbb{R}_+^n :\ \mu_{\boldsymbol{\alpha}}\left(\boldsymbol{x}\right) \leqslant \mu_{\boldsymbol{\beta}}\left(\boldsymbol{x}\right)\right] \Leftrightarrow \boldsymbol{\alpha} \leqslant \boldsymbol{\beta}$ (Muirhead inequality) [1];
b) for $n = 3$ $(\alpha,\, \beta \in \mathbb{N}^+)$

$$\mu_{(\alpha+2\beta,\, 0,\, 0)}\left(\boldsymbol{x}\right) + \mu_{(\alpha,\, \beta,\, \beta)}\left(\boldsymbol{x}\right) \geqslant 2\mu_{(\alpha+\beta,\, \beta,\, 0)}\left(\boldsymbol{x}\right)$$

(Schur inequality [2]).

Theorem. Let $\alpha,\, \beta,\, \gamma \in \mathbb{N}^+$.

1. If $\alpha + \gamma \geqslant 2\beta$ then $\mu_\alpha\left(\boldsymbol{x}\right) + \mu_\gamma\left(\boldsymbol{x}\right) \geqslant 2\mu_\beta\left(\boldsymbol{x}\right).$
2. For $n = 2$,

$$\mu_{(m,\, m-\alpha)}\left(\boldsymbol{x}\right) + \mu_{(m,\, m-\gamma)}\left(\boldsymbol{x}\right) \geqslant 2\mu_{(m,\, m-\beta)}\left(\boldsymbol{x}\right)$$

if and only if $\beta\left(m - \beta\right) \geqslant \left(\beta - \gamma\right)\left(\beta + \gamma - m\right).$

3. For $n = 3$,

$$\mu_{(\alpha+2\beta+2\gamma,\, 0,\, 0)}\left(\boldsymbol{x}\right) + \mu_{(\alpha,\, 2\beta,\, 2\gamma)}\left(\boldsymbol{x}\right) \geqslant 2\mu_{(\alpha+\beta+\gamma,\, \beta+\gamma,\, 0)}\left(\boldsymbol{x}\right).$$

References

1. **G.H.Hardy, J.E.Littlewood, G.Pólya** (1952). Inequalities. Cambridge University Press. pp.324.
2. **Finta, Béla** (2015). A Schur Type Inequality for Five Variables. Procedia Technology. 19: 799-801. Doi:10.1016/j.protcy.2015.02.114.