Singularity preserving maps on matrix algebras

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The first result on linear preservers was obtained by Ferdinand Georg Frobenius, who characterized linear maps on complex matrix algebra preserving the determinant.

Let $M_n(\mathbb{F})$ be the $n \times n$ matrix algebra over a field $\mathbb{F}$ and $\mathcal{Y}$ be a subset of $M_n(\mathbb{F})$. We say that a transformation $T\colon \mathcal{Y} \to M_n(\mathbb{F})$ is of a *standard form* if there exist non-singular matrices P, Q such that

$$T(A) = PAQ \quad \text{or} \quad T(A) = PA^TQ \quad \text{for all } A \in \mathcal{Y}. \tag{1}$$

Frobenius [1] proved that if $T\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is linear and preserves the determinant, i. e., $\det(T(A)) = \det(A)$ for all $A \in M_n(\mathbb{C})$, then T is of the standard form (1) with $\det(PQ) = 1$. In 1949 Jean Dieudonné [2] generalized this result for an arbitrary field $\mathbb{F}$. He replaced the determinant preserving condition by the singularity preserving condition and proved the corresponding result for a bijective map T.

In 2002 Gregor Dolinar and Peter Šemrl [3] modified the classical result of Frobenius by removing the linearity and replacing the determinant preserving condition by

$$\det(A+\lambda B) = \det(T(A)+\lambda T(B)) \quad \text{for all } A, B \in M_n(\mathbb{F}) \text{ and all } \lambda \in \mathbb{F} \tag{2}$$

for $\mathbb{F} = \mathbb{C}$. They proved that if $T\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is surjective and satisfies (2), then T is linear and hence is of the standard form (1) with $\det(PQ) = 1$.

Soon after that, Victor Tan and Fei Wang [4] generalized this proof for a field $\mathbb{F}$ with $|\mathbb{F}| > n$ and showed that under the condition (2) the map T is linear even without the surjectivity condition. Moreover, they revealed that if T is surjective, then only two different values of λ are required in (2). To be more precise, if $|\mathbb{F}| > n$ and $T\colon M_n(\mathbb{F}) \to M_n(\mathbb{F})$ is a surjective map satisfying

$$\det(A + \lambda_i B) = \det(T(A) + \lambda_i T(B)) \quad \text{for all } A, B \in M_n(\mathbb{F}) \text{ and } i = 1, 2,$$

where $\lambda_i \neq 0$ and $(\lambda_1/\lambda_2)^k \neq 1$ for $1 \leqslant k \leqslant n-2$, then T is of the standard form (1).

Nevertheless, this result was also further generalized by Constantin Costara [5]. Suppose $|\mathbb{F}| > n^2$ and $\lambda_0 \in \mathbb{F}$. Let $T\colon M_n(\mathbb{F}) \to M_n(\mathbb{F})$ be a surjective map satisfying (2) only for one fixed value of $\lambda = \lambda_0$: $\det(A + \lambda_0 B) = \det(T(A) + \lambda_0 T(B))$ for all $A, B \in M_n(\mathbb{F})$.

Costara obtained that if $\lambda_0 \neq -1$, then such T is of the standard form (1) with $\det(PQ) = 1$.

For $\lambda_0 = -1$, he showed that there exist $P, Q \in GL_n(\mathbb{F})$, $\det(PQ) = 1$, and $A_0 \in M_n(\mathbb{F})$ such that

$$T(A) = P(A + A_0)Q \quad \text{or} \quad T(A) = P(A + A_0)^T Q \qquad \text{for all } A \in \mathcal{Y}.$$

The aim of this work is to relax the condition (2) for T. It has been revealed that if $\mathbb{F}$ is an algebraically closed field, then the conditions on determinant in the above results of Dieudonné or Tan and Wang can be replaced by less restrictive. The following result has been obtained.

Theorem. Suppose $\mathcal{Y} = GL_n(\mathbb{F})$ or $\mathcal{Y} = M_n(\mathbb{F})$, $T\colon \mathcal{Y} \to M_n(\mathbb{F})$ is a map satisfying the following conditions:

- for all $A, B \in \mathcal{Y}$ and $\lambda \in \mathbb{F}$, the singularity of $A + \lambda B$ implies the singularity of $T(A) + \lambda T(B)$;
- the image of T contains at least one non-singular matrix.

Then T is of the standard form (1).

(Note that in the theorem above $\det(PQ)$ possibly differs from 1.)

References

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