Exponentially Subelliptic Harmonic Maps

Yuan Jen Chiang
University of Mary Washington, Fredericksburg, VA, USA
ychiang@umw.edu
Sorin Dragomir
Universita degli Studi della Basilicata, Potenza, Italy
sorin.dragomir@unibas.it

Exponentially harmonic maps were first introduced by Eells and Lemaire [5] in 1990. Exponential wave maps are exponentially harmonic maps on Minkowski spaces, which were first studied by Chiang and Yang [1, 4] since 2007. Firstly, we deal with the critical points of maps $\phi : \mathbb{H}_n \to S^m$ from the Heiserberg group into a sphere with energy $E_1(\phi) = \int_\Omega \exp(\frac{1}{2}||\nabla^H \phi||^2_\theta)\theta \wedge (d\theta)^n$ for domains $\Omega \subset\subset \mathbb{H}_n$ and a contact structure θ on $\mathbb{H}_n$. They are solutions to the 2nd order quasi-linear subelliptic PDE system

$$-\triangle_b \phi^j + 2e_b(\phi)\phi^j + G_\theta(\nabla^H e_b(\phi), \nabla^H \phi^j) = 0, 1 \leq j \leq m+1,$$

and arise through Fefferman's construction, i.e. as base maps $\phi : \mathbb{H}_n \to S^m$ associated to S^1 invariant exponential wave maps $\Phi : C(\mathbb{H}^n) \to S^m$ from the total space of the canonical circle bundle $S^1 \to C(\mathbb{H}^n) \to S^m$ endowed with the Fefferman's metric F_θ. We establish Caccioppoli type estimates

$$\int_{B_{r(x)}} \exp\left(\frac{Q}{2}||\nabla^H \phi||^2_\theta\right)||\nabla^H u||^Q_\theta \theta \wedge (d\theta)^n \leq Cr^\beta \ (0 < \beta < 1)$$

with $Q = 2n+2$ (the homogeneous dimension of $\mathbb{H}^n$), and show that any weak solution $\phi \in \bigcap_{p \geq Q} W^{1,p}_H(\Omega, S^n)$ of finite p-energy $E_p(\phi) < \infty$ for some $p \geq 2Q$ is locally Hölder continuous, i.e. $\phi^j \in S^{0,\alpha}_{loc}(\Omega)$ (Hölder like spaces) for $0 < \alpha \leq 1$, built in terms of the Carnot-Carathéodory metric ρ_θ. The main theorems and results are based on [2]. Secondly, we study exponentially subelliptic harmonic (e.s.h.) maps from a compact pseudo hermitian manifold (M, θ) into a Riemannian manifold (N, h), i.e. C^2 solutions of $\phi : M \to N$ to nonlinear PDE system $\tau_b(\phi) + \phi_* \nabla^H e_b(\phi) = 0$ which are the Euler-Legrange equation of $\delta E_b(\phi) = 0$ with $E_b(\phi) = \int_M \exp(e_b(\phi))\theta \wedge d\theta^n$, where e.s.h. maps arise in a similar way as the first setting. We study the second variation formula and stability of exponentially subelliptic harmonic maps based on [3].

References:

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