The minimality of Sturm-Liouville problems with a boundary condition depending quadratically on the eigenparameter

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We study minimality of root functions of the following Sturm-Liouville problem

$$-y'' + q(x)y = \lambda y,\ 0 < x < 1, \tag{0.1}$$

$$y(0)\cos\beta = y'(0)\sin\beta,\ 0 \le \beta < \pi, \tag{0.2}$$

$$y(1) = (a\lambda^2 + b\lambda + c)y'(1),\ a \ne 0, \tag{0.3}$$

where λ is the spectral parameter, $q(x)$ is a real valued and continuous function on the interval $[0, 1]$, and a, b, c are real. It is known that the eigenvalues of (0.1)-(0.3) form an infinite sequence, accumulating only at $+\infty$, and one of the following cases are possible:

(a) All the eigenvalues are real and simple;

(b) All the eigenvalues are simple and all, except a conjugate pair of non-real, are real;

(c) All the eigenvalues are real and all, except one double, are simple;

(d) All the eigenvalues are real and all, except one triple, are simple.

By constructing the biorthogonal system explicitly, it is possible to show that in the case (a) the system of eigenfunctions with any two eigenfunctions excluded forms a minimal system in space $L_2(0, 1)$. It is also possible to give similar results in the cases (b), (c) and (d). In particular, for the case (b) one can prove that the system of eigenfunctions without the eigenfunction, corresponding to the double eigenvalue, is a minimal system. Similarly, in the case (c) one can prove that the system of eigenfunctions with no excluded functions is a minimal system. Finally, in the case (d) it is possible to prove that the system of eigenfunctions, without the two eigenfunctions, corresponding to non-real eigenvalues, is minimal. These minimality results can then be extended to basis properties in $L_2(0, 1)$ and $L_p(0, 1)$ $(1 < p < \infty)$. But the study of these properties wouldn't be complete if we will not consider similar questions for the system of root functions which contain beside eigenfunctions some associated functions. If the associated functions are not excluded then these minimality properties do not always hold true. There are associated functions which when included make the system of root functions not minimal anymore. In the current work, we will give necessary and sufficient conditions for the system of root functions to be minimal in $L_2(0, 1)$ for all possible choices of the two excluded functions, including the cases when the excluded functions are eigenfunction and when they are associated functions.